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Oscillation for second order nonlinear delay differential equations with impulses

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Abstract: In this paper , we investigate the oscillation of second order nonlinear delay differential equations with impulses of the form

$$\left[r(t)x'(t)\right]' + p(t)x'(t) + Q\left(t, x(t-\delta)\right) = 0, t \geq t_{0,} t \neq t_k, k = 1, 2, \ldots n$$

$$x\left(t_k^{\ +}\right) = g_k\left(x\left(t_k\right)\right),\ x'\left(t_k^{\ +}\right) = h_k\left(x'\left(t_k^{\ +}\right)\right)$$

Key words :Oscillation; delay; second-order;impulses
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I. Introduction

The theory of impulsive differential equations is now being recognized to be not only richer than the corresponding theory of differential equations without impulses, but also represents a more natural frame work for the mathematical modeling of many real work phenomena[1].There are many papers devoted for the oscillation criteria of second-order differential equations without impulses[2-3],with impulse[4-7] .In[8],the authors obtained the asymptotic behavior for the equations

$$\left[r(t)x'(t)\right]' + p(t)x'(t) + Q\left(t, x(t-\delta)\right) = 0, t \geq t_{0,} t \neq t_k, k = 1, 2, \ldots n$$

$$x\left(t_k^{\ +}\right) = g_k\left(x\left(t_k\right)\right),\ x'\left(t_k^{\ +}\right) = h_k\left(x'\left(t_k^{\ +}\right)\right)$$

In this paper we obtain the oscillation of second order nonlinear delay differential equations with impulses of the form

$$\left[r(t)x'(t)\right]' + p(t)x'(t) + Q\left(t, x(t-\delta)\right) = 0, t \geq t_{0,} t \neq t_k, k = 1, 2, \ldots n \quad (1)$$

$$x\left(t_k^{\ +}\right) = g_k\left(x\left(t_k\right)\right),\ x'\left(t_k^{\ +}\right) = h_k\left(x'\left(t_k^{\ +}\right)\right) \quad (2)$$

$$x(t) = \phi(t), x\left(t_0^+\right) = x_0,\quad x'\left(t_0^+\right) = x_0',\ t \in \left[t_0 - \delta, t_0\right] \quad (3)$$

where for every $t_0 \geq 0, \phi \in PC_{t0} = \left\{\phi : [t_0 - \delta, t_0] \to +\infty / \phi\right\}$

A function $x(t)$ is said to be a solution of (1) and (2) satisfying the initial value condition (3) if

(i) $x : \left[t_0 - \delta, \infty\right] \to \square$ satisfies(1.3) for $t_0 - \delta \leq t \leq t_0; t \geq t_{0,} x\left(t_k^+\right), x'\left(t_k^+\right), x\left(t_k^-\right)\ and x'\left(t_k^-\right)$

(ii) $x(t)$ and $x'(t)$ are continuously differentiable for t > t_0 , $t \neq t_k\ \ t \neq t_k + \delta$ and satisfies (1);

(iii) for $t \geq t_{0,} x\left(t_k^+\right), x'\left(t_k^+\right), x\left(t_k^-\right)\ and x'\left(t_k^-\right)$ exist with $x\left(t_k^+\right) = x\left(t_k\right), x'\left(t_k^+\right) = x'\left(t_k\right)$ and satisfy (2)

As is customary, a solution of (1) and (2) is said to be non oscillatory, if it is eventually positive or eventually negative. Otherwise it will be called oscillatory.

Here, we always assume

(H1) $Q(t, x)$ is continuous in $[t_0, \infty)$, $xQ(t, x) > 0\left(x \neq 0\right)\ and\ Q(t, x)/f(x) \geq q(t)\quad (x \neq 0)$ where $q(t)$ is continuous in $[t_0, \infty)$ and $q(t) \geq 0,\ xf(x).0,\ f'(x) \geq k > 0,\ r(t) > 0$.

(H2) $p(t), g_k(x), h_k(x)$ are continuous in R and there exist positive numbers a_k, a_k^*, b_k, b_k^* such that $a_k^* \le \frac{g_k(x)}{x} \le a_k, b_k^* \le \frac{h_k(x)}{x} \le b_k$.

Lemma 1 Let x(t) be a solution of (1) and (2).Suppose that there exists some $T \ge t_0$ *such that* $x(t) > 0, t \ge T$. If (H1) and (H2) are satisfied and

$$\sum_{m=1}^{n-1}\prod_{k=m}^{n-1}\prod_{l=0}^{m-1} a_k b_1^* \int_{t_{m-1}}^{t_m} \exp\left[\int_{t_0}^{u} \frac{r'(s)+p(s)}{r(s)}ds\right]du$$
$$+\prod_{k=0}^{n-1} b_1^* \int_{t_{n-1}}^{t_n}\left[-\int_{t_0}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right]du = +\infty \tag{4}$$

holds, then $x'(t_j) > 0$ *and* $x'(t) > 0$ *for* $t \in (t_k, t_{k+1}]$, *where* $t_k \ge T, k = 1, 2, \ldots n.$

Proof. At first,we prove that $x'(t_k) > 0$ for any $t_k \ge T$. If it is not true ,then there exist some j such that $t_j \ge T, x'(t_j^+) < 0$. Then $x'(t_j^+) = h_j(x'(t_j)) \le b_j^* x'(t_j)$ .By (1),we have

$$x''(t) + \frac{r'(t)+p(t)}{r(t)}x'(t) + \frac{Q(t,x)}{r(t)} = 0,$$

i.e.,

$$\left[x'(t)\exp\left(\int_{t_j}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)\right]' = -\frac{Q(t,x)}{r(t)}\exp\left(\int_{t_j}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right) \le 0,$$

$$t \in (t_j, t_{j+1}] \tag{5}$$

Hence $x'(t)\exp\left(\int_{t_j}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right)$

is decreasing in $(t_j, t_{j+1}]$,

$$x'(t_{j+1})\exp\left(\int_{t_j}^{t_{j+1}}\frac{r'(s)+p(s)}{r(s)}ds\right) \le x'(t_j^+) \le b_j^* x'(t_j),$$

$$x'(t_{j+1}) \le b_j^* x'(t_j)\exp\left(-\int_{t_j}^{t_{j+1}}\frac{r'(s)+p(s)}{r(s)}ds\right).$$

For $t \in (t_{j+1}, t_{j+2}]$, *we have*

$$x'(t_{j+2}) \le b_{j+1}^* b_j^* x'(t_j)\exp\left(-\int_{t_{j+1}}^{t_{j+2}}\frac{r'(s)+p(s)}{r(s)}ds\right).$$

It is easy to show that for any natural number $n \ge 2$

$$x'(t_{j+n}) \le \prod_{j=0}^{n-1} b_{j+k}^* x'(t_j)\exp\left(-\int_{t_{j+1}}^{t_{j+n}}\frac{r'(s)+p(s)}{r(s)}ds\right). \tag{6}$$

Since

$$x^{'}(t)\exp\left(\int_{t_j}^{t}\frac{r^{'}(s)+p(s)}{r(s)}ds\right)$$

is decreasing in $\left(t_j,t_{j+1}\right]$,hence,

$$x'(t)\le x'(t_j^{+})\exp\left(-\int_{t_j}^{t}\frac{r'(s)+p(s)}{r(s)}ds\right),t\in\left(t_j,t_{j+1}\right].$$

Integrating the above inequality from s to t , we have

$$x(t)\le x\left(s\right)+x'(t_j^{+})\int_{s}^{t}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du,\ \ t_j<s\le t\le t_{j+1}.$$

Let $t\to t_{j+1},s\to t_j^{+}$, we get

$$x(t_{j+1})\le x\left(t_j^{+}\right)+x'(t_j^{+})\int_{t_j}^{t_{j+1}}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du,$$

$$\le a_j x\left(t_j\right)+b_j^{*}x'(t_j)\int_{t_j}^{t_{j+1}}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du,$$

$$x(t_{j+2})\le a_{j+1}a_j x\left(t_j\right)+a_{j+1}b_j^{*}x'(t_j)\int_{t_j}^{t_{j+1}}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du+b_{j+1}^{*}b_j^{*}x'(t_j)\int_{t_j}^{t_{j+1}}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du$$

By induction for any natural number n , we have

$$x(t_{j+n})\le\prod_{k=0}^{n-1}a_{j+k}x\left(t_j\right)+x'(t_j)\sum_{m=1}^{n-1}\prod_{k=m}^{n-1}\prod_{l=0}^{m-1}a_{j+k}b_{j+l}^{*}\int_{t_{j+m-1}}^{t_{j+m}}\exp\left(-\int_{t_j}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du+\prod_{k=0}^{n-1}b_{j+k}^{*}\int_{t_{j+n-1}}^{t_{j+n}}\exp\left(-\int_{t_0}^{u}\frac{r'(s)+p(s)}{r(s)}ds\right)du$$

since , $x\left(t\right)>0,x^{'}\left(t_j\right)\ge 0\left(t_j\ge T\right)$, the above inequality is contrary to condition (4).Therefore $x^{'}\left(t_k\right)\ge 0\left(t_k\ge T\right)$ .Because

$x^{'}\left(t\right)\exp\left(\int_{t_j}^{t}\frac{r^{'}(s)+p(s)}{r(s)}ds\right)$ is decreasing in $(t_j,t_{j+1}]$,

$$x^{'}\left(t\right)\exp\left(\int_{t_j}^{t}\frac{r^{'}(s)+p(s)}{r(s)}ds\right)\ge x^{'}\left(t_{j+1}\right)\exp\left(\int_{t_j}^{t_{j+1}}\frac{r^{'}(s)+p(s)}{r(s)}ds\right)\ge 0$$

Hence $x^{'}\left(t\right)\ge 0,\ t\in(t_k,t_{k+1}]$.

The proof of Lemma 1 is complete.

Theorem 1:

If conditions $(1)-(3)$ *hold, then* $a_k^* < 1$ *and there exist* $G(t) \geq 0 (\leq 0), F(t) \geq 0$ *such that the following equa*

$$\sum_{m=1}^{n}\prod_{j=m}^{n} b_j \left(\psi - \frac{ka(t)r(t)}{4}F^2(t)\right)\exp\left[\int_{t_n}^{t}\frac{p(s)}{r(s)} + kF(s)ds\right]dt = +\infty,$$

where

$$\psi(t) - k^{-1}a(t)\left[kq(t) - p(t)G(t) - \left[r(t)G(t)\right]' + r(t)G^2(t)\right],$$

$$a(t) = \exp\left[-2\int_0^t G(u)du\right].$$

Proof.Let x(t) be a nonoscillatory solution of the differential equation (1) and let $T_0 \geq t_0$ be such that

$x(t) \neq 0$ *for all* $t \geq t_0$.*Without loss of generality, we assume that* $x(t) > 0$ *for all* $t \geq T_0$.*In the following, we*

$$v(t) = a(t)r(t)\left[\frac{x'(t)}{f(x(t-\delta))} + \frac{G(t)}{k}\right].$$

Differentiating the equality and making use of (1) and (H1) we get

$$v'(t) \leq -\psi(t) - \frac{p(t)v(t)}{r(t)} - \frac{kv^2(t)}{a(t)r(t)} + \frac{ka(t)r(t)}{4}F^2(t) - \frac{ka(t)r(t)}{4}F^2(t)$$

$$\leq -\left(\psi(t) - \frac{ka(t)r(t)}{4}F^2(t)\right) - \left(kF(t) + \frac{p(t)}{r(t)}\right)v(t). \qquad (7)$$

For all $t \geq T_0, t \neq t_k$, *with* $\psi(t)$ *defined as above, by* (4)

$$v(t)\exp\left[\int_{t_j}^{t}\frac{p(s)}{r(s)} + kF(s)ds\right]'$$

$$\leq -\left(\psi(t) - \frac{ka(t)r(t)}{4}F^2(t)\right)\ \exp\left[\int_{t_j}^{t}\frac{p(s)}{r(s)} + kF(s)ds\right].$$

Integrating from s *to* s_1

$$v(s_1) \leq v(s)\exp\left[\int_{s_1}^{s}\frac{p(s)}{r(s)} + kF(s)ds\right] - \int_{s}^{s_1}\left[\psi(t) - \frac{ka(t)r(t)}{4}F^2(t)\right]$$

$$\times\exp\left[\int_{t_j}^{t}\frac{p(s)}{r(s)} + kF(s)ds\right],$$

$$v(t_k^+) = a(t_k)r(t_k)\left[\frac{x'(t_k^+)}{f(x_k)} + \frac{G(t_k)}{k}\right]. \qquad (8)$$

In (8) let $s_1 = t_1, s_0 = t_0^+$. Then

$$v(t_1^+) \le b_1 \le b_1 v(t_0^+)\exp\left[\int_{t_1}^{t_0} \frac{p(s)}{r(s)} + kF(s)ds\right]$$

$$-\int_{t_0}^{t_1}\left[\psi(t) - \frac{ka(t)r(t)}{4}F^2(t)\right]\exp\left[\int_{t_1}^{t_0} \frac{p(s)}{r(s)} + kF(s)ds\right].$$

By induction, for any natural number n, we have

$$v(t_n^+) \le \prod_{k=1}^{n} b_k v(t_0^+)\exp\left[\int_{t_n}^{t_0} \frac{p(s)}{r(s)} + kF(s)ds\right]$$

$$\times\sum_{m=1}^{n}\prod_{j=m}^{n} b_j\left[\psi(t) - \frac{ka(t)r(t)F^2(t)}{4}(t)\right]\exp\left[\int_{t_1}^{t_0} \frac{p(s)}{r(s)} + kF(s)ds\right].$$

By the condition of theorem 1 and $v\left(t_0^+\right) \ge 0$, the above inequality is impossible. The proof of theorem is completed.

Theorem 2.

Assume that the condition of lemma 1 hold and $f\left(ab\right) \ge f\left(a\right) f\left(b\right)$ *for any* $ab > 0,$ $f\left(a_k^*\right)/b_k \le 1$ *for* $k \ge 1.$*If there exist* $F\left(t\right), G\left(t\right) \ge 0$ *such that*

$$\sum_{m=1}^{n}\prod_{j=m}^{n}\frac{b_j}{f\left(a_j *\right)}\int_{t_{m-1}}^{t_m}\psi(t) - \frac{ka(t)r(t)}{4}F^2(t)\exp\left[\int_{t_n}^{t}\frac{p(s)}{r(s)} + kF(s)ds\right]dt = +\infty,$$

then every solution of 1 *is oscillatory*

Proof. With loss of generality ,we assume that x(t)>0.By Lemma (1) , $x'(t) \ge 0.$

Let

$$v(t) = a(t)r(t)\left[\frac{x'(t)}{f(x)} + \frac{G(t)}{k}\right]$$

Then $v(t) \ge 0, v\left(t_k^+\right) \ge 0$. Relation (1) yields

$$v\left(t_k^+\right) \le \frac{b_k}{f(a_k^*)} v\left(t_k\right).$$

Following the similar way to the proof of theorem 1, the proof is omitted .

Theorem 3.

Suppose the conditions of Lemma 1 *hold and there exists a positive* integer k_0 *such that* $a_k^* > 1$ *for* $k \geq k_0$*.If*

$$\int_{\in}^{+\infty} \frac{du}{f(u)} < +\infty \left(\int_{\infty}^{-\in} \frac{du}{f(u)} > -\infty \right),$$

And

$$\sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \sum_{j=1}^{n-1} \sum_{i=1}^{j} \frac{1}{b_{k+i}} \int_{t_{k+j}}^{t_{k+j+1}} \frac{q(s)}{r(s)} \exp\left[\int_{t_{k+j}}^{t} \frac{r'(u)+p(u)}{r(u)} du \right] ds$$
$$\times \exp\left[-\int_{t_{k+1}}^{t} \frac{r'(u)+p(u)}{r(u)} du \right] + \exp\left[-\int_{t_{k+1}}^{t} \frac{r'(u)+p(u)}{r(u)} du \right]$$
$$\times \exp\left[\int_{t_{k+1}}^{s} \frac{r'(u)+p(u)}{r(u)} du \right] ds = +\infty$$

hold , then every solution of (1) *is oscillatory*

Proof .Without loss of generality ,we assume that x(t) > 0 for all $t \geq T_0, k_0 = 1.$
By (2) we know

$$x'(t_0^+) \geq x'(t_1) \exp\left(\int_{t_0}^{t_1} \frac{r'(s)+p(s)}{r(s)} ds \right) + \int_{t_0}^{t_1} \frac{Q(t,x)}{r(t)}$$
$$\times \exp\left(\int_{t_0}^{t_1} \frac{r'(s)+p(s)}{r(s)} ds \right) dt,$$
$$x'(t_1^+) \geq \frac{x'(t_2^+)}{b2} \exp\left(\int_{t_1}^{t_2} \frac{r'(s)+p(s)}{r(s)} ds \right) + \int_{t_1}^{t_2} \frac{Q(t,x)}{r(t)}$$
$$\times \exp\left(\int_{t_1}^{t} \frac{r'(s)+p(s)}{r(s)} ds \right) dt.$$

By induction for any natural number k , we have

$$x'(t_k^+) \geq \frac{x'(t_{k+1}^+)}{b_{k+1}} \exp\left(\int_{t_k}^{t_{k+1}} \frac{r'(s)+p(s)}{r(s)} ds \right) + \int_{t_k}^{t_{k+1}} \frac{Q(t,x)}{r(t)}$$
$$\times \exp\left(\int_{t_k}^{t} \frac{r'(s)+p(s)}{r(s)} ds \right) dt.$$

From the above and (2) , noting $x'(t_k^+) \geq 0, we\,get,\ for\ t \in (t_k, t_{k+1}],$

$$\int_{x_0}^{+\infty} \frac{du}{f(u)} \geq \sum_{k=0}^{\infty} \int_{t_k}^{t_{k+1}} \sum_{j=1}^{n-1} \sum_{i=1}^{j} \frac{1}{b_{k+i}} \int_{t_{k+j}}^{t_{k+j+1}} \frac{q(s)}{r(s)} \exp\left[\int_{t_{k+j}}^{t} \frac{r'(u)+p(u)}{r(u)} du\right] ds$$
$$\times \exp\left[-\int_{t_{k+1}}^{t} \frac{r'(u)+p(u)}{r(u)} du\right] + \exp\left[-\int_{t_{k+j}}^{t} \frac{r'(u)+p(u)}{r(u)} du\right]$$
$$\times \int_{t}^{t_{k+1}} \frac{q(s)}{r(s)} \exp\left[\int_{t_{k+1}}^{s} \frac{r'(u)+p(u)}{r(u)} du)\right] ds.$$

This contradicts the hypothesis.

Remark

When r(t)=1,p(t)=0, δ =0,if we take F(t)=G(t)=0,the results of this paper become the ones of those given in [4,5]

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