The Value Distribution of Some Differential Polynomials

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Abstract: We prove a value distribution theorem for meromorphic functions having few poles from which we obtain several interesting results which improve some results of W. Doeringer, C.C.Yang, A.P.Singh, G.P.Barker and others.

I. Introduction And Main Results

Let \(f\) be a transcendental meromorphic function in the plane.

A monomial in \(f\), is an expression of the form \(M[f] = \prod_n \left(f^{(n)}\right)^{n_k}\) where \(n_0, n_1, n_2,..., n_k\) are non-negative integers.

\[ \gamma_M = n_0 + n_1 + n_2 + ... + n_k \]

is called the degree of the monomial and \(\Gamma_M = n_0 + 2n_1 + ... + (k + 1)n_k\) the weight.

If \(M_1[f], M_2[f], ..., M_n[f]\) denote monomials in \(f\), then, \(Q[f] = a_1M_1[f] + a_2M_2[f] + ... + a_nM_n[f]\), where \(a_i \neq 0 (i = 1, 2,..., n)\), is called a differential polynomial in \(f\) of degree \(\gamma_Q = \max \{\gamma_{M_i} : 1 \leq j \leq n\}\) and weight \(\Gamma_Q = \max \{\Gamma_{M_j} : 1 \leq j \leq n\}\)

Also, we call the numbers \(\gamma_Q = \min \{\gamma_{M_i}\}\) and \(k\) (the order of the highest derivative of \(f\)) the lower degree and the order of \(Q[f]\) respectively. If \(\gamma_Q = 0\), \(Q[f]\) is called a homogeneous differential polynomial.

II. Theorem A

W.K.Hayman in his well known problem book ‘Problems in Function Theory’ has raised some interesting open problems related to the value distribution of differential polynomials. In 1988, Hong-Xun Yi[6] proved the following result:

**Theorem A:** Let \(f\) be transcendental meromorphic function in the plane and \(Q_1[f] \neq 0\), \(Q_2[f] \neq 0\) be differential polynomials in \(f\).

Let \(P_1[f] = a_n f^n + a_{n-1} f^{n-1} + ... + a_0 (a_n(z) \neq 0)\) and \(\gamma_{Q_1} = \gamma_{Q_2}\).

If \(F = P_1[f] Q_1[f] + Q_2[f]\), then

\[
\left(n - \gamma_{Q_2}\right) T(r,f) \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{P_1[f]}) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right) \overline{N}(r,f) + S(r,f).
\]

Thinking on the same lines, we considered a different combination of differential polynomials and obtained an interesting result which generalizes the result of Hong-Xun Yi.

The following theorem is our main result.

**Theorem 1:** Let \(f\) be transcendental meromorphic function in the plane and \(Q_1[f] \neq 0\), \(Q_2[f] \neq 0\) be differential polynomials in \(f\).

Let \(P_1[f] = a_n f^n + a_{n-1} f^{n-1} + ... + a_0 (a_n(z) \neq 0)\) \hspace{1cm} (1)

\(P_2[f] = b_m f^m + b_{m-1} f^{m-1} + ... + b_0 (b_m(z) \neq 0)\) \hspace{1cm} (2)

where \(n > m\).

If \(F = P_1[f] Q_1[f] + P_2[f] Q_2[f]\) \hspace{1cm} (3)
Then, \((n - \gamma_Q) T(r, f) \leq N(r, Q) = N\left(\frac{1}{F}, \frac{1}{P_1[f]}\right) + 3N\left(\frac{1}{P_2[f]}\right) + (\Gamma_Q - \gamma_Q) N(r, f) + S(r, f)\).

To prove the above Theorem, we require the following Lemmas.

**Lemma 1** [2]: Suppose \(P[f]\) is as in (1). Then, \(m(r, P[f]) = n m(r, f) + S(r, f)\).

**Lemma 2** [4]: Let \(f(z)\) be a transcendental meromorphic function, \(P[f]\) and \(Q[f]\) be differential polynomials in \(f\). If degree of \(Q[f]\) is at most \(n\) and \(f^n P[f] = Q[f]\), then \(m(r, \frac{Q[f]}{f^n}) = S(r, f)\).

**Lemma 3** [3]: If \(P[f]\) is a homogeneous differential polynomial of degree \(n\), then,
\[
m\left(\frac{r, P[f]}{f^n}\right) = S(r, f).
\]

**Lemma 4** [6]: Suppose that \(Q[f]\) is a differential polynomial in \(f\). Let \(z_0\) be a pole of \(f\) of order \(m\) and not a zero or a pole of the co-efficients of \(Q[f]\). Then \(z_0\) is a pole of \(Q[f]\) of order atmost \(m \gamma_Q + (\Gamma_Q - \gamma_Q)\).

**Lemma 5** [11]: If \(Q[f]\) is a differential polynomial in \(f\) with arbitrary meromorphic co-efficients \(Q_j\), then
\[
m(r, Q[f]) \leq \gamma_Q m(r, f) + \sum_{j=1}^n m(r, Q_j) + S(r, f).
\]

**Proof of Theorem 1:**
We have \(F = P_1[f] Q_1[f] + P_2[f] Q_2[f]\).

Therefore, \(F' = F' P_1[f] Q_1[f] + F' P_2[f] Q_2[f]\).

Also, \(F' = P_1[f] (Q_1[f])' + Q_1[f] (P_1[f])' + P_2[f] (Q_2[f])' + Q_2[f] (P_2[f])'\).

Hence, we have
\[
F' \left[ P_1[f] Q_1[f] + P_2[f] Q_2[f] \right] = P_1[f] (Q_1[f])' + Q_1[f] (P_1[f])' + P_2[f] (Q_2[f])' + Q_2[f] (P_2[f])'.
\]

Therefore,
\[
P_1[f] \left[ F' Q_1[f] - (Q_1[f])' \frac{P_1[f]}{P_2[f]} \right] = P_2[f] \left[ (Q_2[f])' - F' Q_2[f] \frac{P_2[f]}{P_1[f]} \right].
\]

Or, \(P_1[f] \left[ \frac{F'}{P_2[f]} Q_1[f] - (Q_1[f])' \frac{P_1[f]}{P_2[f]} \right] = P_2[f] \left[ (Q_2[f])' - F' Q_2[f] \frac{P_2[f]}{P_1[f]} \right],\)

which is of the form, \(P_1[f] Q_1^*[f] = Q[f]\). (4)

where, \(Q_1^*[f] = \frac{F'}{F P_2[f]} Q_1[f] - (Q_1[f])' \frac{P_1[f]}{P_2[f]} - \frac{P_2[f]}{P_1[f]} Q_1[f]\)

and \(Q[f] = (Q_2[f])' - F' Q_2[f] + \frac{P_2[f]}{P_1[f]} Q_2[f].\) (5)

Without loss of generality, let us assume that \(Q[f] \neq 0\).

By Lemma 2, \(m(r, Q_1^*[f]) = S(r, f)\).

Again from (4), \(P_1[f] = \frac{Q[f]}{Q_1^*[f]}\).
Therefore, 

$$m(r, P_1[f]) \leq m(r, Q[f]) + m\left(r, \frac{1}{Q^*[f]}\right)$$  \hspace{1cm} (6)$$

Again by Lemma 5 and(5), 

$$m(r, Q[f]) \leq \gamma_{Q_2} m(r, f) + S(r, f)$$  \hspace{1cm} (7)$$

From the First Fundamental Theorem, we have

$$m\left(r, \frac{1}{Q^*[f]}\right) = N(r, Q^*[f]) - N\left(r, \frac{1}{Q[f]}\right) + S(r, f)$$  \hspace{1cm} (8)$$

Also, all the poles of $Q^*[f]$ occur at the zeros of $F$.  $P_1[f]$ and $P_2[f]$, the pole of $f$, and the zeros and poles of the co-efficients. Suppose $z_0$ is a pole of $f$ of order $m$. Then $z_0$ is a pole $P_1[f]$ of order $mn$. 

From Lemma 4, $z_0$ is a pole of $Q[f]$ of order atmost $\gamma_{Q_2} + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)$

If $z_0$ is a pole of $Q^*[f]$, since $Q^*[f] = \frac{Q[f]}{P_1[f]}$, 

$z_0$ is a pole of $Q^*[f]$ of order atmost $\gamma_{Q_2} + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right) - mn

= \Gamma_{Q_2} - \gamma_{Q_2} + 1 - m(n - \gamma_{Q_2})$ 

If $z_0$ is a pole of $Q^*[f]$ then since $\frac{1}{Q^*[f]} = \frac{P_1[f]}{Q[f]}$, 

$z_0$ is a zero of $Q^*[f]$ of order atleast $\gamma_{Q_2} + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right) - \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)

Thus, we have,

$$N(r, Q^*[f]) - N\left(r, \frac{1}{Q^*[f]}\right) \leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{P_1[f]}\right) + 3N\left(r, \frac{1}{P_2[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)N(r, f) - (n - \gamma_{Q_2})N(r, f) + S(r, f)$$  \hspace{1cm} (9)$$

In view of (7), (8), (9), the equation (6) becomes,

$$nm(r, f) \leq \gamma_{Q_2} m(r, f) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{P_1[f]}\right) + 3N\left(r, \frac{1}{P_2[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)N(r, f) - (n - \gamma_{Q_2})N(r, f) + S(r, f)$$

Hence, we have

$$\left(n - \gamma_{Q_2}\right)\Gamma(r, f) \leq N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{P_1[f]}\right) + 3N\left(r, \frac{1}{P_2[f]}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)N(r, f) + S(r, f)$$

Hence the result.

Putting $P_1[f] = f^n$ and $P_2[f] = f^{n-1}$ in the above theorem, we get the following.

**Theorem 2:** Let $f$, $Q_1[f], Q_2[f]$ be as defined in Theorem 1.

Let $F = f^n Q_1[f] + f^{n+1} Q_2[f]$.

Then,

$$\left(n - \gamma_{Q_2}\right)\Gamma(r, f) \leq N\left(r, \frac{1}{F}\right) + 4N\left(r, \frac{1}{F}\right) + \left(\Gamma_{Q_2} - \gamma_{Q_2} + 1\right)N(r, f) + S(r, f)$$

If $P_2[f] \equiv 1$, we get the following.

**Theorem 3:** Let $P_1[f], Q_1[f]$ and $Q_2[f]$ be as defined in Theorem 1.
If \( F = P_n[f]Q_1[f] + Q_2[f] \), then
\[
(n - \gamma_{Q_2})T(r, f) \leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}\left(r, \frac{1}{P_n[f]}\right) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f)
\]
which is the result of Hong Xun Yi [5].

As an application of theorem 2, we have the following.

**Theorem 4:** Let \( F = f^n Q[f] \) where \( Q[f] \) is a differential polynomials in \( f \).
If \( n \geq 1 \), then \( \rho_F = \rho_f \) and \( \lambda_F = \lambda_f \)

**Proof:** If \( F = f^n Q[f] \), then by theorem 2, we have
\[
nT(r, f) \leq \bar{N}(r, \frac{1}{F}) + 4\bar{N}(r, \frac{1}{F}) + \bar{N}(r, f) + S(r, f)
\]
Clearly, the zeros and poles of \( f \) are that of \( F \) respectively.

Therefore, \( 4\bar{N}(r, \frac{1}{F}) + \bar{N}(r, f) \leq 4\bar{N}(r, \frac{1}{F}) + \bar{N}(r, f) + S(r, f) \)

Therefore, \( nT(r, f) \leq \bar{N}(r, \frac{1}{F}) + 4\bar{N}(r, \frac{1}{F}) + \bar{N}(r, f) + S(r, f) \)
\[
\leq 6T(r, F) + S(r, f)
\]
Therefore, \( T(r, f) = O(T(r, F)) \) as \( r \to \infty \).

Also, we know that \( T(r, F) = O(T(r, f)) \) as \( r \to \infty \).

Hence the Theorem.

**Theorem 5:** No transcendental meromorphic function \( f \) with \( \bar{N}(r, f) + \bar{N}(r, \frac{1}{F}) = S(r, f) \)
can satisfy an equation of the form
\[
F = a_1 f^n Q_1[f] + a_2 f^{n-1} Q_2[f] + a_3 = 0
\quad (9)
\]
where \( a_1 \neq 0, a_3 \neq 0 \), \( n \) is a positive integer with \( n > \gamma_{Q_2} \), \( Q_1[f] \) and \( Q_2[f] \) are differential polynomials in \( f \).

**Proof:** Suppose there exists a transcendental meromorphic function \( f \) satisfying (9).
Then by theorem 2, we have,
\[
(n - \gamma_{Q_2})T(r, f) \leq \bar{N}(r, \frac{1}{F}) + 4\bar{N}(r, \frac{1}{F}) + (\Gamma_{Q_2} - \gamma_{Q_2} + 1)\bar{N}(r, f) + S(r, f)
\]
\[
\leq 5\bar{N}(r, \frac{1}{F}) + S(r, f)
\]
Or, \( n - \gamma_{Q_2} \leq \frac{5\bar{N}(r, \frac{1}{F}) + S(r, f)}{T(r, f)} = \frac{S(r, f)}{T(r, f)} \) by hypothesis.

Or, \( n - \gamma_{Q_2} \leq 0 \), which implies \( n \leq \gamma_{Q_2} \) which contradicts the choice of \( n \).

Hence the result.

This improves our earlier result namely

**Theorem A [10]:** No transcendental meromorphic function \( f \) with \( N(r, f) = S(r, f) \) can satisfy the equation
\[
a_1(z)f^n(z)\pi_1(f) + a_2(z)f^{n-1} \pi_2(f) + a_3(z) = 0.
\]
where \( \pi_1(f) \) and \( \pi_2(f) \) are differential polynomials of degree \( n \) and \( n - 1 \) respectively and \( n > 1 \), and \( \text{Max}\{\deg(f^{n-1} \pi_2(f))\} < n \) and \( a_1(z) \neq 0 \).
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