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A Note on applications of q-Theory

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***Abstract**: This paper deals with describing application of q-theory in different fields of mathematics and future areas where its use can be extended .*
***Keywords:** q-analogue, q-function, q-hypergeometric function*

I. Introduction

Basic hypergeometric series are called *q*-analogues (basic analogues or *q*-extensions) of hypergeometric series.

q-hypergeometric series, are q-analog generalizations of generalized hypergeometric series, and are in turn generalized by elliptic hypergeometric series. A series x_n is called hypergeometric if the ratio of successive terms x_{n+1}/x_n is a rational function of n. If the ratio of successive terms is a rational function of q^n, then the series is called a basic hypergeometric series. The number q is called the base or parameter which lies between 0 and 1.Value of q determines accuracy of analogue of any classical function.

1.1 Basic Differentiation operator

$$D_{q,x}\, f(x)= \frac{f(qx)-f(x)}{x(q-1)} \tag{1}$$

1.2 q-Integration

$$\int_a^b f(x)dqx = (1-q)\{b\sum_{r=0}^{\infty} q^r\ f(q^r b) - a\sum_{r=0}^{\infty} q^r\ f(q^r a)\} \tag{2}$$

1.3 q-exponential function

A whole family of q exponential function can be defined as

$$E(q,\beta;x)= \sum_{r=0}^{\infty} x^r\ q^{\beta r(r-1)}/[r;q]! \tag{3}$$

depending upon value of β i.e. β=0, β=1/2 and β=1/4 repectively.

II. q-analogue of some statistical functions

2.1 Forward Differences

If $y_0,y_1,y_2,\ldots,yn$ denote a set of values of y, where y=f(z),then $y_1-y_0,y_2-y_1,\ldots,y_n-y_{n-1}$ are called differences of y, where Δ is called forward difference operator and ∇ is called backward difference operator and as 0<q<1, f(z)>f(qz).

$$\Delta f(z) = f(z) - f(qz) \tag{4}$$
$$\Delta^2 f(z) = \Delta f(z) - \Delta f(qz) = f(q^2z) - 2f(qz) + f(z) \tag{5}$$
$$\Delta^3 f(z) = f(q^3z) - 3f(q^2z) + 3f(qz) - f(z) \tag{6}$$
$$\Delta^4 f(z) = f(q^4z) - 4f(q^3z) + 6f(q^2z) - 4f(qz) + f(z) \tag{7}$$

2.2 Backward Differences

$$\nabla f(qz) = f(qz) - f(z) \tag{8}$$
$$\nabla f(q^2z) = f(q^2z) - f(qz) \tag{9}$$
$$\nabla^2 f(q^2z) = \nabla f(q^2z) - \nabla f(qz) = f(q^2z) - 2f(qz) + f(z) \tag{10}$$
$$\nabla^3 f(q^3z) = f(q^3z) - 3f(q^2z) + 3f(qz) - f(z) \tag{11}$$

2.3 q -analogue of Moment Generating Function

Let X be a random variable.

Then ,$EXPEC\left(E_q(tx)\right) = EXPEC\left(\sum\frac{(tx)^r}{[r;q]!}\right) = 1 + t\mu_1 + \frac{t^2}{[2;q]!}\mu_2 + \cdots + \frac{t^r}{[r;q]!}\mu_r,$ (12)

where EXPEC is expected value.

The coefficient of $\frac{t^r}{[r;q]!}$ in the expression is μ_r, the r[th] moment of X about origin.

MGF for discrete random variable with probability distribution

X:	x1 x2……………xn
P(x):	p1 p2……………..pn

$$M(t) = EXPEC\big(E(q,tx)\big) = \sum E_q\,(tX)p_i \qquad (13)$$

MGF for a Continuous Random Variable

Let X be a continuous random variable with probability density function f(x) $,-\infty < x < \infty$

$$M(t) = Expec\big(E(q,tx)\big) = \int_{-\infty}^{\infty} \boldsymbol{E(q,tx)f(x)d(qx)} \qquad (14)$$

Expec(E(q,tx)) is mgf about origin.
Expec(E(q,t(x-a)) is mgf about point a.
$\mathrm{Expec}(\mathrm{E}(\mathrm{q}, \mathrm{t}(\mathrm{x} - \overline{\mathrm{x}}))$ is mgf about origin.

Properties of mgf

$$Expec\left(E\big(q,t(x+y)\big)\right) = Expec\big(E(q,tx)\big) + Expec\big(E(q,ty)\big) \qquad (15)$$

$$Expec\left(E_q\big(t(x+y)\big)\right) = Expec\left(E_q(tx)\right) + Expec\left(E_q(ty)\right) \qquad (16)$$

$$Expec\left(E_q\big(t(u+c)\big)\right) = E_q(tu) + Expec(E_q(tc)) \qquad (17)$$

2.4 q -Distribution Function

If $F_q(x) = \int_{-\infty}^{\infty} f(x)d(qx) = P(X \le x)$ then the function $F_q(x)$ is the probability that the value of the variable will be less or equal to x. Thus, $F_q(x) = P(X \le x)$ and
$F_q(b) - F_q(a) = \int_a^b f(x)d(qx) = P(a \le X \le b)$. $F_q(x)$ is called the cumulative distribution function of X or simply distribution function.

Properties

$$F_q(-\infty) = 0 \text{ and } F_q(\infty) = 1 \qquad (18)$$

2.5 q- analogue of Differential Equation

Solution of second order linear differential equation with constant coefficients

$$D_q^2 y - a_1 D_q^1 y + a_2 = 0 \qquad (19)$$

If auxiliary equation has real and distinct roots m_1 and m_2, general solution is

$$y = A\,E_q(m_1\mathrm{x}) + \mathrm{B}E_q(m_2\mathrm{x}) \qquad (20)$$

or

$$y = A\,E(q; m_1\mathrm{x}) + \mathrm{BE}\,(q; m_2\mathrm{x}) \qquad (21)$$

or

$$y = A\,E(1/q; m_1\sqrt{q}\mathrm{x}) + \mathrm{BE}\,(1/q; m_2\sqrt{q}\mathrm{x}) \qquad (22)$$

Real and equal roots
If m1=m2=m
y= (Ax+B) E_q(mx) or y= (Ax+B) E(q;m$\sqrt{qx}$) or y= (Ax+B) E(1/q;m$\sqrt{qx}$) (23)
Complex Conjugate Roots
If α and β be the real and imaginary parts of the roots then general solution will take form

$$y = C1E\big(q;\ \alpha\sqrt{qx}\big)\sin(q;(\ \beta x + C2) \qquad (24)$$

2.6 Basic Analogue of Integral Transforms

2.6.1 q-Laplace Transform

$$f(s) = \int_0^{\infty} F(t)E(q,-st)d(qt) \qquad (25)$$

2.6.2 q-Fourier Transform

$f(s) = \int_{-\infty}^{\infty} F(t)E(q, -ist)d(qt)$ (26)

2.6.3q-MellinTransform

$f(s) = \int_{0}^{\infty} F(t)t^{s-1}d(qt)$ (27)

2.6.4 q-HankelTransform

$f(s) = \int_{0}^{\infty} F(t)tJ_n\,(st)d(qt)$ (28)

2.7 Basic Analogue of Newton Cotes Integration

I=$\int_a^b w(x)f(x)dqx = \sum_{k=0}^{n} m_k\, f(x_k)$ where $x_1,x_2,\ldots,x_n$ are nodes distributed within limits of integration.

$R_n = (1-q)\{b\sum_{r=0}^{\infty} q^r\ w(q^r b)f(q^r b) - \ a\sum_{r=0}^{\infty} q^r\ w(q^r a)f(q^r a)\} - \sum_{k=0}^{n} m_k\, f(x_k)$ (29)

R_n is the error term.

If w(x) =1 and nodes x_k^s are distributed in [a,b] with x_0 = a , x_n=b and h =(b-a)/n, $x_k = x_0 + kh$, k ∈ n.

$(1-q)\{b\sum_{r=0}^{\infty} q^r\ w(q^r b)f(q^r b) - a\sum_{r=0}^{\infty} q^r\ w(q^r a)f(q^r a)\} = \frac{(b-a)[f(a)+f(b)]}{[2;q]!}$ (30)

where $\frac{(b-a)}{[2;q]!} = h.$

By putting w(x)=1 ,n=1,f(x)=x in $\int_a^b w(x)f(x)dqx = \sum_{k=0}^{n} m_k\, f(x_k)$ we get,

$m_0 = m_1 = \frac{(b-a)}{1+q}$

$\int_a^b w(x)f(x)dqx = \frac{(b-a)}{[2;q]!}(f(a)+f(b))$ (31)

which is analogue of Trapezoidal Rule.

$R_n = \frac{-h^3}{12} D_q^2 f(\xi)$

$= \frac{-h^3}{12q\xi^2(q-1)^2}(f(q^2\xi) - [2;q]f(q\xi) + qf(\xi)]$ (32)

$= \frac{-h^3}{12}\frac{[q_2 f(q_1^2\xi) + q_1 f(q_2^2\xi) - q_1 f(q_1\, q_2\xi) - q_2 f(q_1\, q_2\xi)]}{(q_1-q_2)^2\xi^2}$

By putting n=2 and n=3 we can easily get analogue for Simpson's 1/3 and Simpsons's 3/8 rule.

2.7.1 q-Simpson's 1/3 Rule

By putting w(x)=1 ,n=2,f(x)=x^2 in $\int_a^b w(x)f(x)dqx = \sum_{k=0}^{n} m_k\, f(x_k)$ (33)

we get,

$m_0 = \frac{(b-a)}{[3;q]}, m_1 = \frac{4(b-a)}{[3;q]}, m_2 = \frac{(b-a)}{[3;q]}$ (34)

2.7.2 q-Simpsons's 3/8 Rule

By putting w(x)=1,n=3,f(x)=x^3in$\int_a^b w(x)f(x)dqx = \sum_{k=0}^{n} m_k\, f(x_k)$ (35)

we get,

$m_0 = \frac{3(b-a)}{2[4;q]}, m_1 = \frac{9(b-a)}{2[4;q]}, m_2 = \frac{9(b-a)}{2[4;q]}, m_3 = \frac{3(b-a)}{2[4;q]}$ (36)

For a method of order m

Error = $R_n = \frac{c}{[m+1;q]!} D_q^n f(\xi),$ (37)

where a< ξ <b

Error terms in Trapezoidal Rule, Simpson's 1/3 Rule, Simpson's 3/8Rule

$E_{trp} = -\frac{h^3}{12} D_q^2 f(\xi), E_{smp\ 1/3} = -\frac{Ch^4}{24} D_q^3 f(\xi), E_{smp\ 3/8} = -\frac{3h^5}{80} D_q^4 f(\xi)$ (38)

Weights for Newton's Cotes Integration methods when q tends to one.

n	m_0	m_1	m_2	m_3
0	1/2	1/2		
1	1/3	4/3	1/3	
2	3/8	9/8	9/8	3/8

2.8 q-analogue of Lobatto Integration

$$\int_{-1}^{1} f(x)dqx = 2(1-q)\sum_{r=0}^{\infty} q^r f(q^r) = m_0 f(-1) + m_n f(n) + \sum_{k=1}^{n-1} m_k f(x_k) \quad (39)$$

For n=2 , $\int_{-1}^{1} f(x)dqx = \frac{1}{1+q+q^2}[f(-1)+f(1)+4f(0)]$ (40)

2.9 q- analogue of Radau Integration

$$\int_{-1}^{1} f(x)dqx = 2(1-q)\sum_{r=0}^{\infty} q^r f(q^r) = m_0 f(-1) + \sum_{k=1}^{n} m_k f(x_k) \quad (41)$$

For n=2

$$\int_{-1}^{1} f(x)dqx = \frac{2}{3(1+q+q^2)} f(-1) + \frac{16+\sqrt{6}}{6(1+q+q^2)} f\left(\frac{1-\sqrt{6}}{5}\right) + \frac{16-\sqrt{6}}{6(1+q+q^2)} f\left(\frac{1+\sqrt{6}}{5}\right) \quad (42)$$

Predictor Correctors Method

2.10 q- analogue of Milne's Method

$$y_4 = y_0 + h[4f_0 + 16\frac{\Delta_q f_0}{[2;q]} + \frac{32[2;q]-8[3;q]}{[2;q][3;q]}\Delta_q^2 f_0 + \frac{2}{3}\Delta_q^3 f_0 \left\{\frac{[2;q][3;q]8^4 - 3*16^2[4;q][2;q]+32[4;q][3;q]}{[4;q][2;q][3;q]}\right\}] \quad (43)$$

2.11 q-analogue of Moultons Method

$$y_1 = y_0 + h[f_0 + \frac{\nabla_q f_0}{[2;q]} + \frac{1}{2}(\frac{1}{[3;q]} + \frac{1}{[2;q]})\nabla_q^2 f_0 + \frac{1}{6}\nabla_q^3 f_0 \left(\frac{1}{[4;q]} + \frac{3}{[3;q]} + \frac{2}{[1;q]}\right)] \quad (44)$$

where,Δ_q^n *is K. Conrad*[4] *difference operator*

$$\Delta_q^n = \{(E-q^{n-1}) \dots (E-q)(E-1)\} \quad (45)$$

where E(f(x))=f(x+h)

III. Conclusion

q-analogue of these methods provide an alternate method of solving classical problems where value of q determines the accuracy of result. q-analogue of different transformations can be used in boundary value problems of differential equations as well as in computer problems where parameters play important role .

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