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Numerical evaluation of fractional integrals and derivatives of analytic function

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Abstract: *A numerical technique based on the Cauchy integral formula of complex analysis has been employed for evaluating the integrals and derivatives of fractional orders of an analytic function. The method of subtraction of singularity has been used to evaluate the contour integrals for a desired degree of accuracy.*
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I. Introduction

Fractional calculus has found immense applications in diverse branches of science and engineering (cf. Dalir and Bashour [1], Oldham and Spanier [2] etc.). Fractional calculus has also been utilized for theoretical research work in complex analysis so far as univalent and multivalent functions are concerned (cf. Srivastava and Owa [3], Srivastava and Mishra[4], Patel and Mishra[5], Liu and Patel [6] etc.). Recently Acharya et al[7] have formulated a technique using the central difference operator and have found out numerical approximation of fractional derivatives of analytic functions by evaluating fractional integrals using quadrature methods. Acharya et al [7] have considered the Riemann and Liouville (R-L) definition of fractional integrals and derivatives which is as follows.

$$D^{\beta} f(z) = \frac{1}{\Gamma(-\beta)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1+\beta}} d\zeta \qquad (1)$$

where the parameter β lies in the open interval $(-1,0)$ for fractional integral and is in the open interval $(0,1)$ for fractional derivative. Here $f(\zeta)$ is an analytic function in a domain containing the directed line segment from 0 to z and β is the order of the integral or derivative.

The objective of the present paper is to formulate a method for evaluating numerically the fractional integral and fractional derivative (jointly known as differintegrals of arbitrary order) of analytic functions using the definition based on the celebrated Cauchy integral formula of complex analysis which is regarded as a modern approach by Miller and Ross[8]. The technique devised has been subjected to numerical testing in respect of some standard analytic functions.

II. Formulation Of The Method

An alternative definition for fractional integrals and derivatives given by Osler[9] and found in Miller and Ross[8] based on the Cauchy integral formula is the following.

$$D^{\beta} f(z) = \frac{\Gamma(1+\beta)}{2\pi i} \int_{C} \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+\beta}} \qquad (2)$$

where C is a closed contour, surrounding the point z, contained in the domain of analyticity of the function $f(\zeta)$ and β is not a negative integer, the contour C starts and ends at $\zeta = 0$. It is noteworthy that for non integer values of β the integrand in equation (2) has a branch point at $\zeta = z$. Deforming the contour C into a contour C' lying on both sides of the branch line for $(\zeta - z)^{-\beta-1}$ and a small loop at z, the differintegral $D^{\beta} f(z)$ given by equation (2) reduces to the differintegral given by equation (1). Hence the following equation is evident.

$$\frac{\Gamma(1+\beta)}{2\pi i} \int_{C} \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+\beta}} = \frac{1}{\Gamma(-\beta)} \int_{0}^{z} \frac{f(\zeta)}{(z-\zeta)^{1+\beta}} d\zeta \qquad (3)$$

where $-1 < \beta < 0$ for fractional integral and $0 < \beta < 1$ for fractional derivative.

Let z be a complex number and the contour C be a square γ of side $\sqrt{2}|z|$ having vertices at the following set of points.

$$z_j = |z|(1 - i^{j-1}), j = 1,2,3,4. \qquad (4)$$

Then equation (2) leads to the following equation:

$$D^{\beta}f(z) = \frac{\Gamma(1+\beta)}{2\pi i}\int_{\gamma} \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+\beta}}. \tag{5}$$

Since the contour γ can be expressed as $\gamma = \bigcup_{k=1}^{4} L_k$, where L_k is a directed line segment from z_k to z_{k+1}, $z_5 = z_1$ we have from equation (5) the following equation:

$$D^{\beta}f(z) = \frac{\Gamma(1+\beta)}{2\pi i}\sum_{k=1}^{4}\int_{L_k} \frac{f(\zeta)d\zeta}{(\zeta - z)^{1+\beta}}\,. \tag{6}$$

Each of the integrals under the summation sign in equation (6) can be approximated by an open quadrature rule meant for integral of an analytic function $\varphi(\zeta)$ along a directed line segment L from the point $z_0 - H$ to $z_0 + H$. In general such a rule with *n* nodes is prescribed as follows.

$$Q(\varphi, n, L) = H\sum_{j=1}^{n} w_j \varphi(z_0 + Ht_j) \tag{7}$$

where the quantities t_j are real or imaginary. Some examples of $Q(\varphi, n, L)$ are $Q(\varphi, 3, L)$, $Q(\varphi, 4, L)$ and $Q(\varphi, 5, L)$ which are due to Lether[10], Acharya-Mohapatra[11] and Tosic[12].

It is noteworthy that by direct application of the open quadrature rule $Q(\varphi, n, L)$ to the four integrals in the right hand side of equation (6) the accuracy of the computed value of the differintegral $D^{\beta}f(z)$ is quite likely to be hampered due to the presence of the branch point *z* arising from the factor $(\zeta - z)^{1+\beta}$ in the denominator. To safeguard the accuracy of computation it is essential to use corrective factors while using the quadrature method.

III. Construction Of Corrective Factors

Subtracting $F(\zeta) = \sum_{k=0}^{m} f^{(k)}(z)(\zeta - z)^k/k!$ from $f(\zeta)$ in the numerator of the integrand in the right hand side of equation (6) leads to the following equation.

$$D^{\beta}f(z) = \frac{\Gamma(1+\beta)}{2\pi i}\sum_{k=1}^{4}\int_{L_k} \frac{f(\zeta) - F(\zeta)}{(\zeta - z)^{1+\beta}}d\zeta + \frac{\Gamma(1+\beta)}{2\pi i}\int_{\gamma} \frac{F(\zeta)}{(\zeta - z)^{1+\beta}}d\zeta \tag{8}$$

The second contour integral in the right hand side of equation(9) is denoted as $C(\beta, m)$ and is the corrective factor. In view of equations (5) and (6) $C(\beta, m)$ is same as $D^{\beta}F(z)$. Therefore we have the following:

$$C(\beta, m) = \frac{1}{\Gamma(\beta)}\sum_{k=0}^{m} \frac{(-1)^k z^{k+\beta} f^{(k)}(z)}{k!\,(k+\beta)} \tag{9}$$

Therefore the final form of the approximation for the differintegral of order β is as follows.

$$\left.\begin{aligned} D^{\beta}f(z) &= \frac{\Gamma(1+\beta)}{2\pi i}\sum_{k=1}^{4} Q(\psi, n, L_k) + C(\beta, m), \\ \psi(\zeta) &= \{f(\zeta) - F(\zeta)\}/(\zeta - z)^{1+\beta} \end{aligned}\right\} \tag{10}$$

IV. Numerical Tests

Using the three rules Q_3, Q_4, Q_5 due to Lether[9], Acharya and Mohapatra[10] and Tosic[11] computations have been made for *n*=3,4,5 for the functions e^{ζ} and $sin\zeta$. The values of $|Error|$ in computing the differintegrals have been appended in Table-I.

It is noteworthy that the value of *m* for which accuracy of fourteen or more decimal places is obtained 9. All computations have been performed considering $z = (1+i)/10$. It is observed that the accuracy of the computed values of fractional integrals is more than that of fractional derivatives by considering either larger value of *m* in the corrective factor or larger value of *n*, the order of the quadrature rule the accuracy of the computed values can be enhanced.

TABLE-I

Function	β	$\lvert Error \rvert$ in Q_3	$\lvert Error \rvert$ in Q_4	$\lvert Error \rvert$ in Q_5
e^{ζ}	-1/2	5.55 (-17)	5.55 (-17)	1.00 (-16)
	-1/4	1.11 (-16)	1.11 (-16)	1.14 (-16)
	1/2	9.03 (-16)	6.68 (-16)	6.68 (-16)
	1/4	9.93 (-16)	1.11 (-15)	1.11 (-15)
$sin\zeta$	-1/2	3.46 (-18)	1.43 (-17)	3.46 (-18)
	-1/4	0.0	1.96 (-17)	0.0
	1/2	6.20 (-17)	1.24 (-16)	1.24 (-16)
	1/4	1.66 (-16)	3.60 (-16)	3.60 (-16)

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