IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 9, Issue 2 (Nov. – Dec. 2013), PP 38-46
www.iosrjournals.org

Comparison of Adomian decomposition and Homotopy perturbation methods for higher-order linear fractional integro-differential equations

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Abstract: *In this paper, we will compare between Adomian decomposition method (ADM) and Homotopy perturbation method (HPM) for obtaining the numerical solutions of higher-order linear fractional integro-differential equations with boundary conditions. Numerical examples are presented to illustrate the efficiency and accuracy of the proposed methods.*
Keywords: *Adomian decomposition method, Homotopy perturbation method, Boundary value problems, Fractional integro-differential equations, Caputo fractional derivative.*

I. Introduction

Fractional differential equations have attracted much attention which provides an efficient for the description of many practical dynamical phenomena arising in engineering and scientific disciplines such as, physics, biology, chemistry, economy, electrochemistry, electromagnetic, control theory, viscoelasticity, see [1-6]. Many mathematical formulations of physical phenomena lead to integro-differential equations such as, fluid dynamics, continuum and statistical mechanics, see [7- 11].
In this paper, we considered the linear boundary value problems for higher-order fractional integro-differential equations with a Caputo fractional derivative of the type:

$$D^{\alpha}y(x)=f(x)+\gamma y(x)+\lambda\int_0^x k(x,t)y(t)dt,\ 0<x<b,\ m-1<\alpha\le m,\ m\in Z^{+}, \tag{1}$$

subject to the following boundary conditions:

$$y(0)=\gamma_0,\qquad y^{(i)}(0)=\gamma_i, \tag{2}$$

$$y(b)=\beta_0,\qquad y^{(i)}(b)=\beta_i, \tag{3}$$

where D^{α} is a caputo fractional derivative, $\gamma_0,\ \gamma_i,\ \beta_0,\ \beta_i,\ \lambda$ and γ are real constants $(i=2k,\ k\in Z^{+},\ 1\le k< m/2)$, $f(x)$ and $k(x,t)$ are given and can be approximated by Taylor polynomials. The existence and stability of solutions for fractional integro-differential equations [12- 14]. He [15- 19] was the first to propose the Adomian decomposition method (ADM) and homotopy perturbation method (HPM) for finding the solutions of non-linear problems. Most fractional integro-differential equations do not have exact solutions, so must we use the approximate techniques. There are many methods for seeking approximate solutions such as variational iteration method, homotopy perturbation method, homotopy analysis method, the fractional differential transform method and Adomian decomposition method, see[20- 23].The outline of this paper is as follows: In section 2,we present some preliminaries. Section 3, contains the application of the Adomian decomposition method. Section 4, contains the application of the homotopy perturbation method. Finally, Section 5, devoted to illustrate some numerical examples on mentioned methods.

II. Preliminaries

Definition 2.1. A real function $f(x),\ x>0,$ is said to be in the space $C_{\alpha},\ \alpha\in R,$ if there exists a real number $p>\alpha,$ such that $f(x)=x^{p}f_1(x),$ where $f_1(x)\in C[0,\infty).$

Definition 2.2. A real function $f(x),\ x>0,$ is said to be in the space $C_{\alpha}^{k},\ k\in N,$ if $f^{k}\in C_{\alpha}.$

Definition 2.3. I^{α} denotes the fractional integral operator of order α in the sense of Riemann-Liouville, defined by:

$$I^{\alpha} f(x)=\begin{cases}\dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x \frac{f(t)}{(x-t)^{1-\alpha}}\,dt, & \alpha>0,\\ f(x), & \alpha=0.\end{cases} \tag{4}$$

Definition 2.4. Let $f\in C_{-1}^{m},\ \mathrm{m}\in N$. Then the Caputo fractional derivative of $f(x)$, defined by:

$$D^{\alpha} f(x)=\begin{cases}\dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\,dt, & 0\le m-1<\alpha\le m,\\ \dfrac{d^m f(x)}{dx^m}, & \alpha=m\in N.\end{cases} \tag{5}$$

Now, we introduce some basic properties of fractional operator are listed below [1]:

For $f\in C_{\mu},\ \mu\ge -1,\ \gamma\ge -1,\ \alpha,\beta\ge 0$:

$$\left.\begin{aligned}
&(1) I^{\alpha} I^{\beta} f(x)=I^{\alpha+\beta} f(x)=I^{\beta} I^{\alpha} f(x).\\
&(2) I^{\alpha} x^{\gamma}=\frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}x^{\alpha+\gamma}.\\
&(3) D^{\alpha}\left[I^{\alpha} f(x)\right]=f(x).\\
&(4) I^{\delta}\left[D^{\alpha} f(x)\right]=f(x)-\sum_{k=0}^{m-1} f^{k}(0)\frac{x^k}{k!},\ \ 0\le m-1<\alpha\le m\in N.
\end{aligned}\right\} \tag{6}$$

III. Adomian decomposition method

Consider the equation (1) with boundary conditions (2), (3). Applying the integral operator I^{α} to both sides of the equation (1), we get:

$$y(x)=\sum_{j=0}^{i}\gamma_j\frac{x^j}{j!}+I^{\alpha}\left[f(x)+\gamma y(x)\right]+I^{\alpha}\left[\lambda\int_0^x k(x,\ t)\,y(t)\,dt\right], \tag{7}$$

according to the Adomian decomposition method [15, 16], we put the solution $y(x)$ be decomposed by infinite series of components as follows:

$$y(x)=\sum_{n=0}^{\infty} y_n(x). \tag{8}$$

Substitution the decomposition (8) into both sides of (7), we get:

$$\sum_{n=0}^{\infty} y_n(x)=\sum_{j=0}^{i}\gamma_j\frac{x^j}{j!}+I^{\alpha}\left[f(x)+\gamma\sum_{n=0}^{\infty} y_n(x)\right]+I^{\alpha}\left[\lambda\int_0^x k(x,\ t)\sum_{n=0}^{\infty} y_n(t)dt\right]. \tag{9}$$

From equation (9), the iterations are determined as follows:

$$y_0(x)=\sum_{j=0}^{i}\gamma_j\frac{x^j}{j!}+I^{\alpha}\left[f(x)\right], \tag{10}$$

$$y_{n+1}(x)=\gamma I^{\alpha}\left[y_n(x)\right]+I^{\alpha}\left[\lambda\int_0^x k(x,\ t)\,y_n(t)\,dt\right],\ n\ge 0. \tag{11}$$

Where $\gamma_1=y'(0),\ \gamma_3=y'''(0),\ \cdots$ and $\gamma_i=y^{(i)}(0)$ are to be determined. The decomposition series solutions are generally converging very rapidly [24-29], we approximate the series solution of ADM by the following $N-$terms truncated series:

$$\Phi_N(x)=y_0(x)+y_1(x)+y_2(x)+\cdots+y_{N-1}(x). \tag{12}$$

Substitution (3) in (12), we get the following system of equations:

$$\left.\begin{aligned}
&y_0(b)+y_1(b)+y_2(b)+\cdots+y_{N-1}(b)=\beta_0,\\
&y_0''(b)+y_1''(b)+y_2''(b)+\cdots+y_{N-1}''(b)=\beta_2,\\
&\vdots\\
&y^{(i)}{}_0(b)+y^{(i)}{}_1(b)+y^{(i)}{}_2(b)+\cdots+y^{(i)}{}_{N-1}(b)=\beta_i.
\end{aligned}\right\} \tag{13}$$

From the system of equations (13), we can find the unknowns $\gamma_1,\ \gamma_3,\ \ldots,\ \gamma_i$. Substitution the constant values of $\gamma_1,\ \gamma_3,\ \ldots,\ \gamma_i$ in equation (12), we get the approximate solution of the problem (1) - (3).

IV. Homotopy perturbation method

Consider the equation (1) with boundary value conditions (2), (3). According to HPM [17 - 19], we construct the following homotopy:

$$(1-P)\mathrm{D}^{\alpha}\, y(x)+P\left(\mathrm{D}^{\alpha}\, y(x)-f(x)-\gamma y(x)-\lambda\int_0^x k(x,\ t)\,y(t)dt\right)=0, \tag{14}$$

or

$$\mathrm{D}^{\alpha}\, y(x)=P\left(f(x)+\gamma y(x)+\lambda\int_0^x k(x,\ t)\,y(t)dt\right), \tag{15}$$

where $P\in[0,1]$ is an embedding parameter. If $P=0$, then equation (15) becomes a linear equation,

$$D^{\alpha}y(x)=0, \tag{16}$$

and when $P=1$, then the equation (15) becomes the original equation (1). The solution of equation (1) can be written as a power series in P as follows:

$$y(x)=y_0(x)+Py_1(x)+P^2y_2(x)+P^3y_3(x)+\cdots\cdots \tag{17}$$

Put $P=1$ in equation (17), so the approximate solution of equation (1) is:

$$y(x)=y_0(x)+y_1(x)+y_2(x)+y_3(x)+\cdots\cdots \tag{18}$$

The convergence of the series (18) is proven in [30]. Substituting (17) in (15), then equating the terms with identical power of P , we obtain the series of linear equations:

$$P^0\ :D^{\alpha}\ y_0(x)=0, \tag{19}$$

$$\left.\begin{aligned}
&P^1:\mathrm{D}^{\alpha}\ y_1(x)= f(x)+\gamma y_0(x)+\lambda\int_0^x k(x,\ t)\,y_0(t)dt,\\
&P^2:\mathrm{D}^{\alpha}\ y_2(x)=\gamma y_1(x)+\lambda\int_0^x k(x,\ t)\,y_1(t)dt,\\
&P^3:\mathrm{D}^{\alpha}\ y_3(x)=\gamma y_2(x)+\lambda\int_0^x k(x,\ t)\,y_2(t)dt,\\
&\vdots
\end{aligned}\right\} \tag{20}$$

From equation (19), the initial approximation can be chosen as follows:

$$y_0(x)=\sum_{j=0}^{i}\gamma_j\frac{x^j}{j!},\quad \left(i=2k,\ k\in Z^+,\ 1\le \mathrm{k}<m/2\right) \tag{21}$$

where $\gamma_1=y'(0),\ \gamma_3=y'''(0),\ \cdots$ and $\gamma_i=y^{(i)}(0)$ are to be determined by applying boundary conditions (3). Equation (19) and system of equations (20) can be solved by applying the integral operator I^{α}, and then by using simple computation, we approximate the series solution of HPM by the following $N-$terms truncated series:

$$\Phi_N(x)=y_0(x)+y_1(x)+y_2(x)+\cdots+y_{N-1}(x). \tag{22}$$

Substitution (3) in (22), we get the following system of equations:

$$\left.\begin{aligned}
&y_0(b)+y_1(b)+y_2(b)+\cdots+y_{N-1}(b)=\beta_0,\\
&y_0''(b)+y_1''(b)+y_2''(b)+\cdots+y_{N-1}''(b)=\beta_2,\\
&\vdots\\
&y^{(i)}{}_0(b)+y^{(i)}{}_1(b)+y^{(i)}{}_2(b)+\cdots+y^{(i)}{}_{N-1}(b)=\beta_i.
\end{aligned}\right\} \tag{23}$$

From the system of equations (23), we can find the unknowns $\gamma_1,\ \gamma_3,\ \ldots,\ \gamma_i$. Substitution the constant values of $\gamma_1,\ \gamma_3,\ \ldots,\ \gamma_i$ in equation (22), we get the approximate solution of the problem (1)-(3).

V. Numerical examples

In this section we will apply ADM and HPM for higher-order fractional integro-differential equations with known exact solutions at $\alpha = 4,\ \alpha = 6.$ All results are obtained by using Maple 16:

Example1. Consider the following linear fourth-order fractional integro-differential equation:

$$D^{\alpha} y(x) = 1 - (1+x)e^{-x} - \int_0^x e^{-t} y(t)\,dt,\ 0 < x < 1,\ 3 < \alpha \le 4, \tag{24}$$

subject to the following boundary conditions:

$$y(0) = 0, \qquad y''(0) = 0 \tag{25}$$

$$y(1) = 1, \qquad y''(1) = 0. \tag{26}$$

For $\alpha = 4,$ the exact solution of the above problem (24) - (26) is $y(x) = x.$

According to ADM, the recursive Adomian decomposition algorithm is:

$$y_0(x) = \sum_{j=0}^{3} \gamma_j \frac{x^j}{j!} + I^{\alpha}\left[f(x) \right], \tag{27}$$

$\vdots$

$$y_{n+1}(x) = I^{\alpha}\left[-\int_0^x e^{-t} y_n(t)\,dt \right]. \tag{28}$$

Where $A = y'(0),\ B = y'''(0)$ is to be determined. To avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term: e.g. $e^{\pm x} = 1 \pm x + x^2/2! \pm x^3/3!.$ Thus, by solving (27), (28), we can form the $N-$terms approximation, $N = 2$:

$$y_0(x) = Ax + \frac{Bx^3}{6} + \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{2x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)},$$

$$\Phi_2(x) = y_0(x) + I^{\alpha}\left[-\int_0^x e^{-t} y_0(t)\,dt \right]. \tag{29}$$

Where $A,\ \mathrm{B}$ can be determined by using boundary condition (26) in $\Phi_2(x)$ (see Table 1).

Table1. Values of A and B for different values of α using (29).

	$\alpha = 3.25$	$\alpha = 3.50$	$\alpha = 3.75$	$\alpha = 4$
A	1.001289323	1.000824834	1.000521904	1.000326805
B	–0.008770162045	–0.005565630077	–0.003495472122	–0.002173843410

We compute the absolute error functions $E_1(x) = |x - \Phi_{2,\,3.25}|,\ E_2(x) = |x - \Phi_{2,\,3.50}|$ and $E_3(x) = |x - \Phi_{2,\,3.75}|.$ Where x is the exact solution of (24)-(26) and $\Phi_{2,\,3.25},\ \Phi_{2,\,3.50}$ and $\Phi_{2,\,3.75}$ are approximate solutions of (24) - (26) by using (29) at $\alpha = 3.25,\ \alpha = 3.50$ and $\alpha = 3.75$ respectively.

According to HPM, we construct the following homotopy:

$$D^{\alpha} y(x) = P\left(1 - xe^{-x} - e^{-x} - \int_0^x e^{-t} y(t)\,dt \right). \tag{30}$$

Substituting (17) in (30), we obtain the following series of linear equations with identical power of P :

$$P^0 : D^{\alpha}\, y_0(x) = 0, \tag{31}$$

$$\left.\begin{aligned} &P^1 : \mathrm{D}^{\alpha}\, y_1(x) = 1 - xe^{-x} - e^{-x} - \int_0^x e^{-t} y_0(t)\,dt, \\ &P^2 : \mathrm{D}^{\alpha}\, y_2(x) = -\int_0^x e^{-t} y_1(t)\,dt, \\ &\vdots \end{aligned}\right\} \tag{32}$$

Applying the operator I^{α} to the above series of linear equations and using the initial condition (25), we get:

$$y_0(x) = 0, \tag{33}$$

$$\left.\begin{aligned}
&y_1(x) = Ax + \frac{Bx^3}{6} + I^\alpha\left[1 - xe^{-x} - e^{-x} - \int_0^x e^{-t} y_0(t)\,dt\right],\\
&y_2(x) = I^\alpha\left[-\int_0^x e^{-t} y_1(t)\,dt\right],\\
&\vdots
\end{aligned}\right\} \tag{34}$$

Where $A = y'(0),\ B = y'''(0)$ is to be determined. To avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term in the system (34): e.g. $e^{\pm x} = 1 \pm x + x^2/2! \pm x^3/3!$. Thus, by solving (33), (34), we obtain $y_1, y_2, \ldots$.

$$y_1(x) = Ax + \frac{Bx^3}{6} + \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{2x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)}, \tag{35}$$

$\vdots$

Now, we can form the $N-$terms approximation, $N = 2$:

$$\Phi_2(x) = Ax + \frac{Bx^3}{6} + \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{2x^{\alpha+3}}{\Gamma(\alpha+4)} + \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)}. \tag{36}$$

Where A, B can be determined by using boundary condition (26) in $\Phi_2(x)$ (see Table 2).

Table2. Values of A, B for different values of α using (36).

	$\alpha = 3.25$	$\alpha = 3.50$	$\alpha = 3.75$	$\alpha = 4$
A	1.010100244	1.007586266	1.005567011	1.004001323
B	–0.08553725904	–0.06165650476	–0.04373516366	–0.03055555556

We compute the absolute error functions $E_4(x) = |x - \Phi_{2,\,3.25}|$, $E_5(x) = |x - \Phi_{2,\,3.50}|$ and $E_6(x) = |x - \Phi_{2,\,3.75}|$. Where x is the exact solution of (24) - (26) and $\Phi_{2,\,3.25}$, $\Phi_{2,\,3.50}$ and $\Phi_{2,\,3.75}$ are approximate solutions of (24) - (26) by using (36) at $\alpha = 1.25$, $\alpha = 1.50$ and $\alpha = 1.75$ respectively. In Fig.1 we compare the absolute error functions.

Example2. Consider the following linear sixth-order fractional integro-differential equation:

$$D^\alpha y(x) = -1 + (2-x)e^x + \int_0^x t y(t)\,dt,\ 0 < x < 1,\ 5 < \alpha \le 6, \tag{37}$$

subject to the following boundary conditions:

$$y(0) = 1, \qquad y''(0) = 1, \qquad y^{(4)}(0) = 1 \tag{38}$$

$$y(1) = e, \qquad y''(1) = e, \qquad y^{(4)}(1) = e. \tag{39}$$

For $\alpha = 6$, the exact solution of the above problem (37) - (39) is $y(x) = e^x$.

According to ADM, the recursive Adomian decomposition algorithm is:

$$y_0(x) = \sum_{j=0}^{5} \gamma_j \frac{x^j}{j!} + I^\alpha\left[f(x)\right], \tag{40}$$

$\vdots$

$$y_{n+1}(x) = I^\alpha\left[\int_0^x t y(t)\,dt\right]. \tag{41}$$

Where $A = y'(0),\ B = y'''(0)$ and $C = y^{(5)}(0)$ are to be determined. To avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term: e.g. $e^x = 1 + x + x^2/2! + x^3/3!$. Thus, by solving (40), (41), we can form the $N-$terms approximation, $N = 2$:

$$y_0(x) = 1 + Ax + \frac{Bx^3}{6} + \frac{x^4}{24} + \frac{Cx^5}{120} + \frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)},$$

$$\Phi_2(x) = y_0(x) + I^{\alpha}\left[\int_0^x t y_0(t)\,dt\right]. \tag{42}$$

Where A, B and C can be determined by using boundary condition (39) in $\Phi_2(x)$, (see Table 3).

Table3. Values of A, B and C for different values of α using (42).

	$\alpha = 5.25$	$\alpha = 5.50$	$\alpha = 5.75$	$\alpha = 6$
A	0.9987943931	0.9985646077	0.9990890315	1.000033984
B	1.013697556	1.015730408	1.009763351	0.9996955212
C	0.2955891970	0.5610041471	0.7981238648	1.002067372

We compute the absolute error functions $E_7(x) = \left|e^x - \Phi_{2,\,5.25}\right|$, $E_8(x) = \left|e^x - \Phi_{2,\,5.50}\right|$ and $E_9(x) = \left|e^x - \Phi_{2,\,5.75}\right|$. Where e^x is the exact solution of (37) - (39) and $\Phi_{2,\,5.25}$, $\Phi_{2,\,5.50}$ and $\Phi_{2,\,5.75}$ are approximate solutions of (37) - (39) by using (42) at $\alpha = 3.25$, $\alpha = 3.50$ and $\alpha = 3.75$ respectively.

According to HPM, we construct the following homotopy:

$$D^{\alpha} y(x) = P\left(-1 + (2 - x)e^x + \int_0^x t y(t)\,dt\right). \tag{43}$$

Substituting (17) in (43), we obtain the following series of linear equations with identical power of P :

$$P^0 : D^{\alpha} y_0(x) = 0, \tag{44}$$

$$\left.\begin{aligned} &P^1 : \mathrm{D}^{\alpha} y_1(x) = -1 + (2 - x)e^x + \int_0^x t y_0(t)\,dt, \\ &P^2 : \mathrm{D}^{\alpha} y_2(x) = \int_0^x t y_1(t)\,dt, \\ &\vdots \end{aligned}\right\} \tag{45}$$

Applying the operator I^{α} to the above series of linear equations and using the initial condition (38), we get:

$$y_0(x) = 1, \tag{46}$$

$$\left.\begin{aligned} &y_1(x) = 1 + Ax + \frac{Bx^3}{6} + \frac{x^4}{24} + \frac{Cx^5}{120} + I^{\alpha}\left[-1 + (2 - x)e^x + \int_0^x t y_0(t)\,dt\right], \\ &y_2(x) = I^{\alpha}\left[\int_0^x t y_1(t)\,dt\right], \\ &\vdots \end{aligned}\right\} \tag{47}$$

Where $A = y'(0)$, $B = y'''(0)$ and $C = y^{(5)}(0)$ are to be determined. To avoid difficult fractional integration, we can take the truncated Taylor expansions for the exponential term in the system (47): e.g. $e^x = 1 + x + x^2/2! + x^3/3!$. Thus, by solving (46), (47), we obtain y_1, y_2, …

$$y_1(x) = Ax + \frac{x^2}{2} + \frac{Bx^3}{6} + \frac{x^4}{24} + \frac{Cx^5}{120} + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)}, \tag{48}$$

$$\vdots$$

Now, we can form the $N-$terms approximation, $N = 2$:

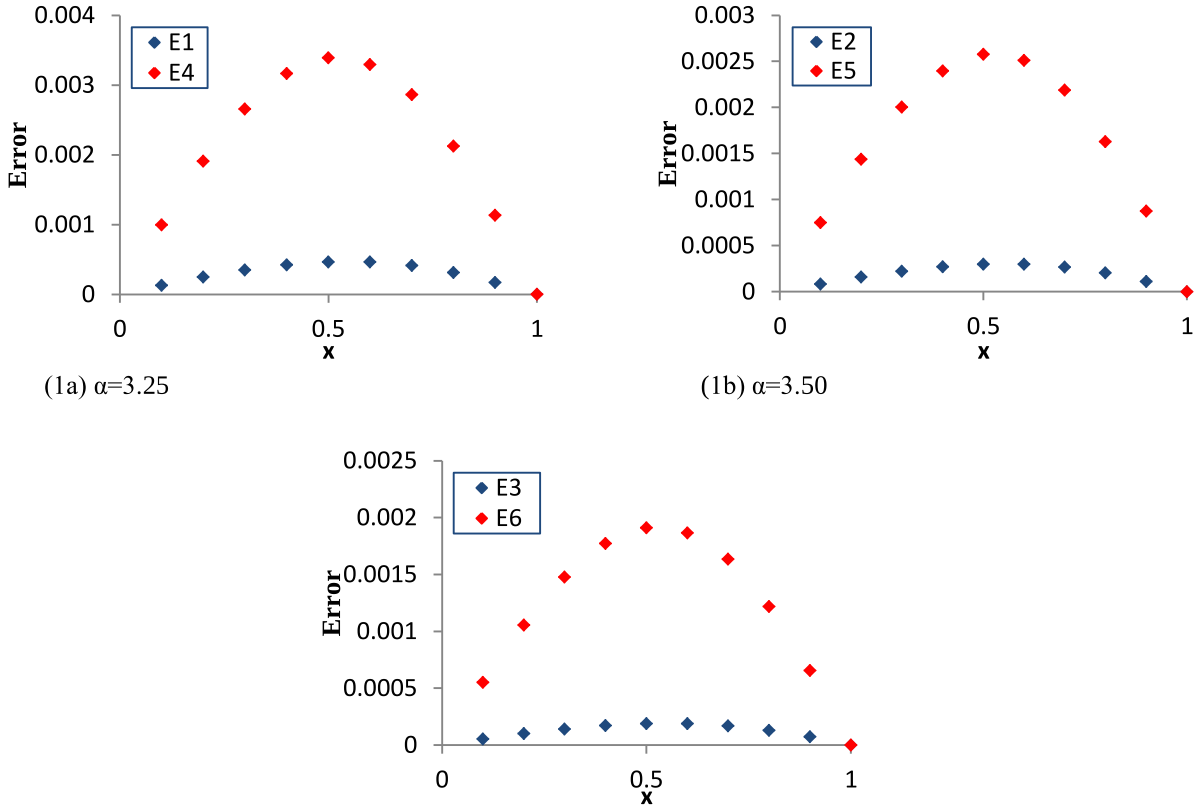
$$\Phi_2(x) = 1 + Ax + \frac{x^2}{2} + \frac{Bx^3}{6} + \frac{x^4}{24} + \frac{Cx^5}{120} + \frac{x^{\alpha}}{\Gamma(\alpha+1)} + \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{x^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{x^{\alpha+3}}{\Gamma(\alpha+4)} - \frac{4x^{\alpha+4}}{\Gamma(\alpha+5)}. \tag{49}$$

Where A, B and C can be determined by using boundary condition (39) in $\Phi_2(x)$ (see Table 4).

Table4. Values of A, B and C for different values of α using (49).

	$\alpha=5.25$	$\alpha=5.50$	$\alpha=5.75$	$\alpha=6$
A	0.9999697551	0.9993589934	0.9996189375	1.000383104
B	1.002993635	1.008525151	1.004974548	0.9965509942
C	0.3727393281	0.6121554492	0.8316703803	1.023837383

We compute the absolute error functions $E_{10}(x)=\left|e^x-\Phi_{2,\,5.25}\right|$, $E_{11}(x)=\left|e^x-\Phi_{2,\,5.50}\right|$ and $E_{12}(x)=\left|e^x-\Phi_{2,\,5.75}\right|$. Where e^x is the exact solution of (37) - (39) and $\Phi_{2,\,5.25}$, $\Phi_{2,\,5.50}$ and $\Phi_{2,\,5.75}$ are approximate solutions of (37) - (49) by using (49) at $\alpha=5.25$, $\alpha=5.50$ and $\alpha=5.75$ respectively. In Fig.2 we compare the absolute error functions.

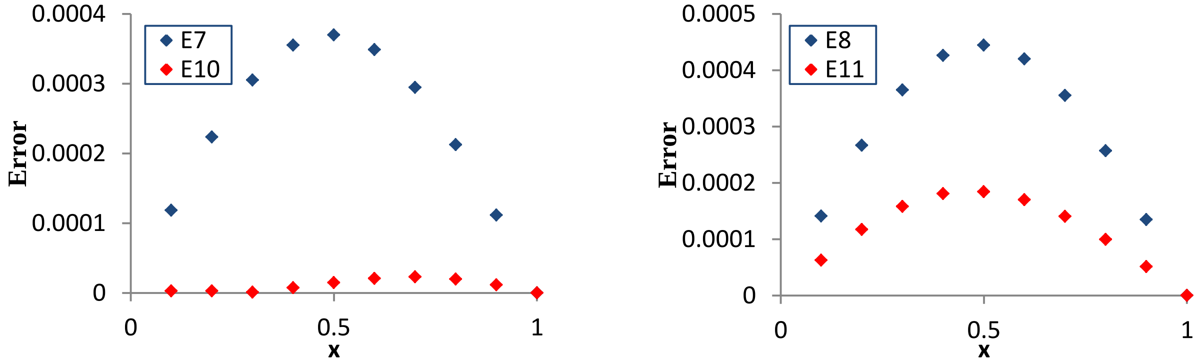


(1a) α=3.25

(1b) α=3.50

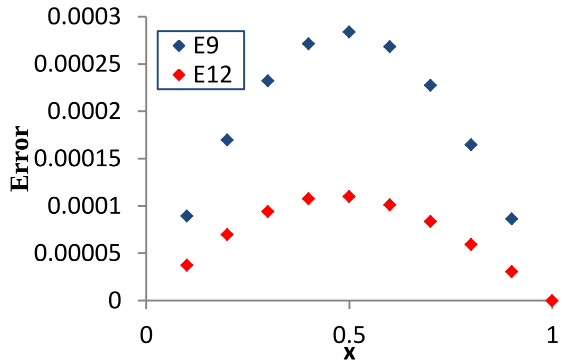
(1c) α=3.75

Fig.1. Comparison of absolute error functions E_1 (x) – E_6 (x) obtained by ADM and HPM for different α.



(2a) α=5.25

(2b) α=5.50



(2c) α=5.75

Fig.2. Comparison of absolute error functins E_7 (x) – E_{12} (x) obtained by ADM and HPM for different α.

VI. Conclusion

In this paper, this study showed that the numerical results of most linear fractional integro-differential equations (1) – (3) as follows:

Case (I): If $\gamma = 0$ and λ is a negative real number, we find ADM is better than HPM (see Fig. 1).

Case (II): If $\gamma = 0$ and λ is a positive real number, we find HPM is better than ADM (see Fig. 2).

Also it is shown that the accuracy can be improved by more $N-$terms of approximated solutions and by taking more terms in the Taylor expansion of the exponential term.

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