IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN:2319-765X. Volume 9, Issue 2 (Nov. – Dec. 2013), PP 29-32
www.iosrjournals.org

Some notes on Second Countablility in Frames

Mary Elizabeth Antony[1], Sabna.K.S.[2] ,Dr.Mangalambal N.R.[3]
[1](Mathematics Dept., Mar Athanasius College/ Mahatma Gandhi University, India)
[2](Mathematics Department, K.K.T.M College/Calicut University , India)
[3](Mathematics Department, St.Joseph'sCollege/Calicut University , India)

Abstract: *In this paper we have tried to develop second countability in frames parallel to that in classical topology.*
Keywords: *$B_L{}^2$ frame, Dense Sublocale , L-base, Locales, Lower frame .*

I. Introduction

With the work of Marshall Stone on the topological representation of Boolean algebras and distributive lattices, the connection between topology and lattice theory began to be explored.
As a dual notion of category of frames we have the category of locales. The category of pointless topological space called locales is an extension of ordinary topological space .

In locales the concept of points gets replaced with "opens". Thus the introduction to the concepts of pointless topology was laid. Most concepts of point set topology like separation axioms ,compactification ,etc… have already be defined into the contexts of locales and proved the analogues theorems.
Here we have tried to define second countability and separability in the contexts of locale theory.

II. Preliminaries

2.1 Frame [1] : A poset $(A,\leq)$ is a frame if and only if
i)Every subset has a join
ii)every finite subset has a meet
iii)binary meets distribute over joins :
$x \wedge \bigvee Y = \bigvee \{x \wedge y : y \epsilon Y\}$ (frame distributivity)
2.2 Boolean Algebra[2]:A Boolean algebra is a distributive lattice A equipped with an additional unary operation $\urcorner : A \rightarrow A$ such that $\urcorner a$ is a complement of a i.e $\urcorner a \wedge a = 0$ and $\urcorner a \vee a = 1$.
2.3 Heyting Algebra[2] : A lattice A is said to be a Heyting algebra ,if for each pair of elements (a,b)there exists an element $(a \rightarrow b)$such that $c \leq (a \rightarrow b)$ iff $c \wedge a \leq b$.
2.4 $A_{\urcorner\urcorner}$ [2]: Given a Heyting Algebra A ,we say $a \epsilon A$ is regular if $\urcorner\urcorner a = a$. The set of all regular elements of A with its induced order ,is denoted $A_{\urcorner\urcorner}$.
2.5 Minimal Elements[3] :Let A be a poset . An element $x \epsilon A$ is said to be a minimal element iff $c \leq a$ implies c=a.
2.6 Frame homomorphism[1] : A function between two frames is a frame homomorphism iff it preserves all joins and finite meets.
2.7 Frm[2]: this is the category of frames whose objects are complete lattices satisfying the infinite distributive law,and whose morphisms are functions preserving the finite meets and arbitrary joins
2.8 Loc[2] : It is the dual of Frm. And the frames are now referred to as Locales.
2.9 Sublocale[3]:A subset $S \subseteq L$ is a sublocale if
i)it is closed under all meets
ii)for every $s \in S$ and every $x \in L$, $x \rightarrow s \epsilon S$.

With the help of these preliminaries, we have tried to develop some definitions and a few theorems corresponding to it.

III. Second countability

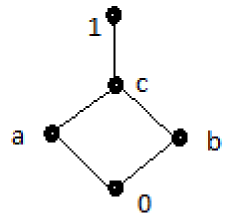
3.1 L-base

Definition : A collection B_L of elements of a locale L is said to be L-base for the locale L if for every $a(\neq 0) \in L$,there exists a non empty sub-collection $\{b_i : b_i \in B, i \in \Delta\}$ such that

$$\bigvee_{i \in \Delta} b_i \leq a \text{ and } \bigvee\{b_i : b_i \in B\} = 1.$$

3.1.1 Examples

3.1.1.1. In the background of frames the class B_L of all singletons form a base for Discrete topology on X
3.1.1.2.For a frame



$B_{L} = \{a,b\}$

3.1.1.3. In the background of frames the collection B_L of compact sets forms a base for locally compact spaces.

3.1.1.4.In the frame of Upper closed sets of finite (say 1,2,3,.....N) list of bits 0,1 ,the collection $B_L = \{\uparrow l : l \text{ is a list of length } N\}$ form L-base

3.1.2 Remark 1

If B_L is a L-base for the frame (locale) L iff $\forall$ p$\in$ L there exists some $b_p \in B_L$ such that $b_p \leq p$.

Proof

Let B_L be a L-base for L then $\forall$ p $\in$ L there exists some $\{b_i : i \in \Delta\}$such that

$$\bigvee b_i \leq p$$

$$\exists\, b_p \leq \bigvee b_i \leq p \text{ ,where } b_p \in \{b_i : i \in \Delta\}.$$

Now suppose $\forall$ p $\in$ L there exists $b_p \in B_L$ such that $b_p \leq p$.Let $x \in \downarrow p$ then $\forall$ x $\in\downarrow$p, by our assumption there exists $b_x \in B_L$ such that $b_x \leq x \leq p$

Therefore $\bigvee b_x \leq p \ \forall\, x \in \downarrow p$.

Thus B_L is a base for the given locale L

3.1.3 Remark 2

Let B_L be a L base for some frame(locale) L $B_L{}^*$ be a class containing B_L ie,

$B_L \subseteq B_L{}^*$ then $B_L{}^*$ is also a L -base for L.

Proof

Let p $\in$ L then by remark 1 we know that there exists $b_p \in B_L$ such that $b_p \leq p$ .Since

$B_L \subseteq B_L{}^*$,$b_p \in B_L{}^*$ implies $b_p \leq p$ for some $b_p \in B_L{}^*$. So by Remark 1 we have $B_L{}^*$ is also a L- base for L .

3.2 Theorem

Let B_L be a collection of elements of L .Then B_L is a L-base for L if for any $b_1, b_2 \in B_L$ there exists some $b_3 \in B_L$ such that $b_3 \leq b_1 \wedge b_2$.

Proof

Let B_L be a L-base then for $b_1, b_2 \in B_L$, $b_1 \wedge b_2 \in$ L (L being a frame).Then ,by the definition of base there exists $b_3 \in B_L$ such that $b_3 \leq b_1 \wedge b_2$.

3.2 Lower frame

3.2.1Definition :Let L be a frame with base B_L and let L* be a frame with base $B_{L*}{}^*$. Then we say that L is a lower frame to L*,denoted as $L \prec_B L^*$ if for every $b \in B_L$

$b = \bigvee\{b^* : b^* \in B_{L*}{}^*\}$.

3.2.1 Example : R with usual topology is a lower frame to R with upper limit topology.

3.3 2-countability

3.3.1 Definiton

A frame L is said to be a $B_L{}^2$ frame if it has a countable L- base. And this property is called 2-countability.

3.3.2 Example

The frame of Upperclosed sets of finite list of bits 0and 1 is second countable.

3.3.3 Theorem

2-countability is hereditary.

Proof

Let L be a $B_L{}^2$ frame. Let h : L→M be a sublocale.

Let B ={ b_1,b2,...} be a countable base for L.

Then we will show that {h(b1),h(b2),,,} is a countable base for M

Let c $\in$ M ,then since h is onto there exists a $\in$ L such that h(a)= c

$a \in L \implies \bigvee b_i \leq a \implies h(\bigvee b_i) \leq h(a)$

$\implies \bigvee h(b_i) \leq h(a) = c$

3.4 Lindeloff

If every cover $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ of a frame L has a countable sub cover then L is said to be Lindeloff.

3.4.1 Theorem

Every 2- countable frame L is Lindeloff.

Proof

Let $\mathcal{B}_L$ be a countable L- base for L .Suppose $\mathcal{U}$ is any cover of L .For each $U \in \mathcal{U}$ there is some $b_{1,U} \in \mathcal{B}_L$ such that $b_{1,U} \leq U$.

Now $\mathcal{B}'=\{ b_{1,U} : U \in \mathcal{U} \}$ is a countable set , since $\mathcal{B}' \subseteq \mathcal{B}_L$, say $\mathcal{B}' = \{b_{i,U1}, b_{2,U2}, b_{3,U3}, ...\}$

Since$\mathcal{B}_L$ is a L base we know that $1 = \bigvee\{b : b \in \mathcal{B}_L\}$ .Then U_1, U_2,..... is a countable sub-cover from $\mathcal{U}$.

3.4.2 Remark

If L is $\mathcal{B}_L^{\,2}$ then clearly the set of all minimal elements is countable.

3.4.3 Proposition [2]

$A_{\neg\neg}$ is a Boolean Algebra

For Proof see [2].

3.4.4 Lemma [2]

Every locale has a smallest dense sublocale , $A_{\neg\neg}$

For Proof see [2]

3.4.5 Dense sublocale[3]

A sublocale S of a locale A is said to be dense if $\bar{S} = \uparrow(\bigwedge S) = A$ .Or equivalently S is dense iff it $0_A \in S$

IV. Separability

As in classical topology, a frame is said to be separable if it has a countable dense sublocale.

4.1 Theroem

Every $\mathcal{B}_L^{\,2}$ frame is separable.

Proof

From the above lemma we know that every locale A has a smallest dense sublocale $A_{\neg\neg}$

So if we can prove that $A_{\neg\neg}$ is countable then we are through.Let $\mathcal{B}_L=\{b_1,b_2,b_3,.....\}$ be a base for the locale A..

If we can prove that $A_{\neg\neg} \subseteq \mathcal{B}_L$, then we are done.

Now let $a \in A_{\neg\neg}$and $a \notin \mathcal{B}_L$.

$\neg b_i \in A_{\neg\neg}\ \forall\ b_i$

$b_1 \not\leq a \wedge \neg b_1$

$b_2 \not\leq a \wedge \neg b_2$

..

..

..

i..e, in general , $b_i \not\leq a \wedge \neg b_i$

∴There exist no element $b_i \in \mathcal{B}_L$ such that $b_i \leq \bigwedge\{ a \wedge \neg b_1, a \wedge b\neg_2, a \wedge \neg b_3............\}$. But this is not possible since $\mathcal{B}_L$ is a base for the sublocale $A_{\neg\neg}$

Thus $a \in \mathcal{B}_L$. Here a was an arbitrary element from $A_{\neg\neg}$,hence $A_{\neg\neg} \subseteq \mathcal{B}_L$.

A being $\mathcal{B}_L^{\,2}$,we have $\mathcal{B}_L$ to be countable ,hence $A_{\neg\neg}$is countable .

Thus A is separable.

4.2 Lemma

Let $h:A \to B$ be a frame homomorphism. If a is a regular element of A, then h(a) is a regular element in B

Proof

Since a is a regular element in A,we have $\neg\neg a=a$

Ie,$a \vee \neg a=1$ and $a \wedge \neg a=0$

Since h is a frame homomorphism we have

$h(a \wedge \neg a)=h(0)=0_B$ and $h(a \vee \neg a)=h(1)=1_B$

ie,$h(a) \wedge h(\neg a)=0_B$ and $h(a) \vee h(\neg a)=1_B$

ie,$h(\neg a)=\neg h(a)$

hence $\neg\neg h(a)=\neg(\neg h(a))=\neg h(\neg a)=h(\neg\neg a)=h(a)$

Therefore ,h(a) is a regular element in B.

4.3Theorem

Homomorphic image of a separable space is separable

Proof

Let A be separable and let $h:A \to B$ is a frame homomorphism.

Since A is s separable , $A_{\neg\neg}$ is a countable dense sublocale of A. Now consider $h(A_{\neg\neg})$.Clearly this is a countable and dense. Hence h(a) is separable.

V. Conclusion

Frame theory finds its applications in computer science where the domain theory provides a mathematical foundation for semantics of programming languages. Here we have developed the concepts of second countability and separability in locale theory. Further study can be done in developing an equivalent theory for local base and first countability

References

Books

[1] Steven Vickers,*Topology via* logic(Cambridge Tracts in Theoretical Computer Science,Cambridge University Press1989)
[2] Peter.T.Johnstone,*Stone Spaces*(Cambridge University Press,1982)
[3] Jorge Picardo, Ales Pultr ,*Frames and Locales –Topology without points*(springer Basel AG 2012)