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q-Iterative Methods

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Abstract: *This paper is devoted to derivation of q-analogues of Iterative Methods for solution of algebraic and transcendental equations and comparing accuracy of results with classical methods.*
Keywords: *Basic, q-special function, q-analogue*

I. Introduction

A q-hypergeometric function is an analogue of an ordinary hypergeometric function with addition of an extra parameter q, where 0<q<1 . When q tends to one , the q-hyper geometric function tends to normal hypergeometric function. Results on basic differentiations, integrations, basic transformations and identities, basic analogue of certain classical functions and their applications are available in the literature .q-analogue finds applications in a number of areas, including the study of fractals and multi-fractal measures, and expressions for the entropy of chaotic dynamical systems. The word *q-analogue* and *basic analogue* are synonyms.

1.1 Basic Number System

$$[\alpha;q]=\frac{1-q^{\alpha}}{1-q}=1+q+q^2+\cdots+q^{(\alpha-1)} \quad (1)$$

1.2 Basic Factorial Notation

$$[n;q]!=[n;q][n-1;q]\ldots[3;q][2;q][1;q] \quad (2)$$

1.3 q-exponential function

$$E_q(x)=\sum_{r=0}^{\infty} x^r/[r;q]! \quad (3)$$
$$E_q^{-1}(x)=\sum_{r=0}^{\infty} x^r\, q^{\frac{r(r-1)}{2}}/[r;q]! \quad (4)$$
$$E(q,x)=\sum_{r=0}^{\infty} x^r\, q^{\frac{r(r-1)}{4}}/[r;q]! \quad (5)$$

1.4 Basic Differentiation operator

$$D_{q,x}f(x)=\frac{f(qx)-f(x)}{x(q-1)} \quad (6)$$

Formula for q differentiation by parts

$$D_{q,x}(f(x)g(x))=g(x)D_{q,x}f(x)+f(x)\,D_{q,x}\,g(x) \quad (7)$$

$$D_{q,x}\frac{f(x)}{g(x)}=\frac{g(x)\,D_{q,x}\,f(x)-f(x)\,D_{q,x}g(x)}{g(x)g(qx)} \quad (8)$$

1.5 Ward-Alsalam [5] q differentiation operator containing two parameters

$$D_{q_1q_2}f(x)=\frac{f(q_1x)-f(q_2x)}{x(q_1-q_2)} \text{ where } q_1=q_2^{-1} \quad (9)$$
$$DD_{q_1q_2}f(x)=\frac{[q_2f(q_1^2x)+q_1f(q_2^2x)-q_1f(q_1\,q_2x)-q_2f(q_1\,q_2x)]}{(q_1-q_2)^2x^2} \quad (10)$$

1.6 Basic analogue of Taylor's Theorem

Jackson(1909) introduced q analogue of Taylor's Theorem

$f(x)=f(a)+\frac{(x-a)^{(1)}}{[1:q]}D_q\ f(a)+\frac{(x-a)^{(2)}}{[2;q]!}\ D_q^2\,f(a)+\ldots+\frac{(x-a)^{(n)}}{[n;q]!}\ D_q^n f(a)$, where

$R_n=\frac{(x-a)^{(n+1)}}{[n+1;q]!}D^{(n+1)}f(\xi)$, where ξ *lies between x and a.* (11)

It can be rewritten as

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)[x-a]_n \quad (12)$$

For double parameter it can be written as

$$f(x) = f(a) + \frac{(x-a)^{(1)}}{[1;q]!} D_{q_1 q_2}\, f(a) + \frac{(x-a)^{(2)}}{[2;q]!}\, D^2_{q_1 q_2} f(a) + \ldots + \frac{(x-a)^{(n)}}{[n;q]!}\, D^n_{q_1 q_2} f(a) \quad (13)$$

where q_1 and q_2 are inverses of each other.

II q-analogue of Iterative Methods

2.1 q-analogue of Newton Raphson Method

$$x_{n+1} = x_n - \frac{f(x_n) x_n (1-q)}{f(x_n) - f(qx_n)} \quad (14)$$

2.2 q-analogue of Newton Raphson Method having multiplicity

$$x_{n+1} = x_n - \quad m \frac{f(x_n) x_n (1-q)}{f(x_n) - f(qx_n)} \quad (15)$$

, where m is multiplicity.

2.3 q-analogue of Newton Raphson Method for multiple root

$$x_{k+1} = x_k - \frac{m(3-m)^1}{[2;q]} \frac{f(k)}{D_q f(k)} - \frac{m^2}{[2;q]} \left(\frac{f(k)}{D_q f(k)}\right)^2 \frac{D_q^2\, f(k)}{D_q f(k)} \quad (16)$$

2.4 q-analogue of Newton Raphson Method for complex root

For complex root Newton Raphson method will look like

$$z_{n+1} = z_n - \frac{f(z_n) z_n (1-q)}{f(z_n) - f(qz_n)} \quad (17)$$

and initial approximation z_o must be complex.

After finding one root z_1, it can be applied on the deflated polynomial

$f^*(z) = \frac{f(z)}{z - z_1}$,If number of roots are k, new iteration should be applied on

$$f^*(z) = \frac{f(z)}{(z-z_1)(z-z_2)\ldots(z-z_k)} \quad (18)$$

So, $z_{n+1} = z_n - \frac{f^*(z_n) z_n (1-q)}{f^*(z_n) - f^*(qz_n)}$ (19)

2.5 q-analogue of Chebyshev Method

$f(x) = f(x_k + x - x_k)$ and approximating f(x) by second degree Taylor's Series Expansion about point x_k we get,

$$x_{k+1} = x_k - \frac{f(k)}{D_q f(k)} - \frac{1}{[2;q]} (x_{k+1} - x_k)^2 \frac{D_q^2\, f(k)}{D_q f(k)} \quad (20)$$

Also, we can write it like

$$x_{k+1} = x_k - \frac{f(k)}{D_q f(k)} - \frac{1}{[2;q]} \left(\frac{f(k)}{D_q f(k)}\right)^2 \frac{D_q^2\, f(k)}{D_q f(k)} \quad (21)$$

By simplifying we can write this equation as

$$x_{n+1} = x_n - \frac{x_n f(x_n)(1-q)}{f(x_n) - f(qx_n)} - \frac{(1-q) x_n f(x_n)[f(q^2 x_n) - [2;q] f(qx_n) + q f(x_n)]}{[2;q] q\, (f(x_n) - f(qx_n))^3} \quad (22)$$

2.6 q-analogue of Newton Raphson Method with two parameters

$$x_{k+1} = x_k - \frac{f(x_k)}{f(q_1 x_k) - f(q_2 x_k)} (q_1 - q_2) x_k \quad (23)$$

2.7 q-analogue of Newton Raphson Method for multiple roots

$$x_{k+1} = x_k - \frac{f(x_k)}{f(q_1 x_k) - f(q_2 x_k)} m (q_1 - q_2) x_k \quad (24)$$

2.8 q-analogue of Chebyshev Method having two parameters

$$x_{k+1} = x_k - \frac{f(k)}{D_q f(k)} - \frac{1}{[2;q]} \left(\frac{f(k)}{D_q f(k)}\right)^2 \frac{D_q^2\, f(k)}{D_q f(k)} \quad (25)$$

$$x_{k+1} = x_k - \frac{f(x_k)(q_1 - q_2) f(x_k)}{f(q_1 x_k) - f(q_2 x_k)} - \frac{1}{[2;q]} \left(\frac{f(x_k)(q_1 - q_2) f(x_k)(q1 - q2)}{f(q_1 x_k) - f(q_2 x_k)}\right)^2 \left(\frac{q_2 f(q_1^2 x) + q_1 f(q_2^2 x) - q_1 f(q_1 q_2 x) - q_2 f(q_1 q_2 x)}{(q_1 - q_2) x_k} \frac{xk}{f(q_1 x_k) - f(q_2 x_k)}\right) \quad (26)$$

2.9 q-analogue Multipoint Iteration Method with two parameters

$$x_{k+1}^{*} = x_k - \frac{f(x_k)}{f(q_1 x_k) - f(q_2 x_k)} \frac{(q_1 - q_2)\, x_k}{[2;q]} \quad (27)$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f(q_1 x_{k+1}^{*}) - f(x_{k+1}^{*}\, q_2)} \frac{(q_1 - q_2) x_{k+1}^{*}}{[2;q]} \quad (28)$$

Second Method

$$x_{k+1}^{*} = x_k - \frac{f(x_k)}{f(q_1 x_k) - f(q_2 x_k)} \frac{(q_1 - q_2)\, x_k}{[1;q]} \quad (29)$$

$$x_{k+1} = x_k - \frac{f(x_{k+1}^{*})}{f(q_1 x_k) - f(x_k q_2)} \frac{(q_1 - q_2)\, x_k}{[1;q]} \quad (30)$$

PROBLEM 1(solution by q method with double parameter)

Let us take a problem $xe^{x} - 1 = 0$, let us take x_0 =1

Table 1: Calculation of x_1 by iterative method (q-analogue of Numerical Methods) for different values of q_1

Value of q_1	Value of q_2	$f(q_1x_0)$	$f(q_2x_0)$	$Dq_1q_2f(x)$	x_1
0.99999999	1.00000001	1.718281774	1.718281883	5.436563675	0.683939722
0.999	1.001001001	1.71284934	1.723727922	5.436569552	0.683940063
0.98	1.020408163	1.611167117	1.830945828	5.438968093	0.684079443
0.97	1.030927835	1.558806126	1.890377286	5.442030897	0.684257245
0.96	1.041666667	1.507228614	1.952016987	5.446388237	0.684509852
0.95	1.052631579	1.456424176	2.015980164	5.452083986	0.684839443
0.94	1.063829787	1.406382533	2.082389756	5.459164858	0.685248225
0.93	1.075268817	1.357093535	2.151377031	5.467680616	0.685738442
0.92	1.086956522	1.308547159	2.223082277	5.477684304	0.686312366
0.91	1.098901099	1.260733505	2.297655556	5.489232495	0.686972299
0.9	1.111111111	1.2136428	2.375257531	5.502385566	0.68772057
0.89	1.123595506	1.16726539	2.45606038	5.517207992	0.688559534
0.88	1.136363636	1.121591742	2.540248803	5.533768677	0.689491569
0.87	1.149425287	1.076612443	2.628021124	5.552141311	0.690519075
0.86	1.162790698	1.032318197	2.719590522	5.57240476	0.691644469
0.85	1.176470588	0.988699824	2.815186379	5.594643501	0.692870184
0.84	1.19047619	0.94574826	2.915055782	5.618948091	0.694198665
0.83	1.204819277	0.903454554	3.019465185	5.645415698	0.695632364
0.82	1.219512195	0.861809867	3.128702256	5.674150669	0.69717374
0.81	1.234567901	0.820805469	3.243077929	5.705265171	0.698825247
0.8	1.25	0.780432743	3.362928697	5.738879898	0.700589338
0.79	1.265822785	0.740683177	3.488619163	5.775124844	0.702468453
0.78	1.282051282	0.701548367	3.620544898	5.814140178	0.704465015
0.77	1.298701299	0.663020015	3.759135637	5.856077201	0.706581425
0.76	1.315789474	0.625089928	3.904858867	5.901099418	0.708820051
0.75	1.333333333	0.587750012	4.05822386	5.949383738	0.711183224
0.74	1.351351351	0.550992281	4.219786205	6.001121803	0.713673229
0.73	1.369863014	0.514808844	4.390152936	6.056521489	0.716292292
0.72	1.388888889	0.479191912	4.569988314	6.115808574	0.719042575
0.71	1.408450704	0.444133794	4.760020382	6.17922863	0.721926161
0.7	1.428571429	0.409626895	4.961048405	6.247049131	0.724945043
0.69	1.449275362	0.375663718	5.173951334	6.319561854	0.728101114
0.68	1.470588235	0.342236858	5.399697452	6.397085573	0.731396148
0.67	1.492537313	0.309339005	5.639355397	6.479969121	0.734831788
0.66	1.515151515	0.276962941	5.894106796	6.56859487	0.738409529
0.65	1.538461538	0.245101539	6.165260763	6.663382676	0.7421307
0.55	1.818181818	-0.04671084	10.20117652	8.080771393	0.787361658
0.54	1.851851852	-0.073356294	10.79927385	8.28800152	0.792678389
0.53	1.886792453	-0.099565876	11.44937879	8.511946423	0.798132913
0.52	1.923076923	-0.125345622	12.15765068	8.754328507	0.80372203
0.51	1.960784314	-0.150701491	12.93117127	9.01710381	0.809441938
0.5	2	-0.175639365	13.7781122	9.302501042	0.815288187

Table 2: Calculation of x_2 by iterative method (q-analogue of Numerical Methods) for different values of q_1

Value of q_1	Value of q_2	$f(q_1x1)$	$f(q_2x1)$	$Dq_1q_2f(x1)$	f(x1)	x2
0.99999999	1.00000001	1.718281665	1.718281774	5.436563495	1.71828172	0.683939711
0.999	1.001001001	0.353062621	0.357629544	3.337016362	0.35534369	0.577454612
0.98	1.020408163	0.310637846	0.402928352	3.338730616	0.35580886	0.577509342

0.97	**1.030927835**	**0.288977125**	**0.428261121**	**3.340919925**	**0.3564024**	**0.577579321**
0.96	**1.041666667**	**0.267764639**	**0.454701682**	**3.344035159**	**0.35724595**	**0.577679052**
0.95	**1.052631579**	**0.246990835**	**0.482316489**	**3.348108305**	**0.35834709**	**0.577809725**
0.94	**1.063829787**	**0.226646219**	**0.511176888**	**3.353173608**	**0.3597136**	**0.577972657**
0.93	**1.075268817**	**0.206721351**	**0.541359552**	**3.359267741**	**0.36135352**	**0.578169309**
0.92	**1.086956522**	**0.187206843**	**0.572946954**	**3.366430002**	**0.36327508**	**0.57840129**
0.91	**1.098901099**	**0.168093349**	**0.606027895**	**3.374702518**	**0.36548679**	**0.57867037**
0.9	**1.111111111**	**0.149371562**	**0.640698081**	**3.384130478**	**0.36799737**	**0.578978494**

Table 3: Calculation of x3 by iterative method (q-analogue of Numerical Methods) for different values of q_1

$f(q_1x_2)$	$f(q_2x_2)$	$Dq_1q_2f(x_2)$	$f(x_2)$	x_3
0.355342497	**0.355342542**	**3.337012114**	**0.35534252**	**0.577454476**
0.027112248	**0.030359438**	**2.810233997**	**0.028734265**	**0.567229746**
-0.00326886	**0.062332853**	**2.81116954**	**0.028888077**	**0.567233165**
-0.01893519	**0.080033774**	**2.812364816**	**0.029084763**	**0.567237575**
-0.03438198	**0.098377586**	**2.81406652**	**0.02936511**	**0.567243936**
-0.04961177	**0.117398702**	**2.8162931**	**0.029732504**	**0.567252406**
-0.06462693	**0.137134175**	**2.819064585**	**0.03019071**	**0.567263179**
-0.07942957	**0.157623957**	**2.82240272**	**0.030743904**	**0.567276497**
-0.09402159	**0.178911181**	**2.826331118**	**0.031396709**	**0.567292645**
-0.10840467	**0.201042479**	**2.830875412**	**0.032154223**	**0.567311967**
-0.12258028	**0.224068332**	**2.836063442**	**0.033022065**	**0.567334867**

Proceeding in this way we get x_4= 0.5671433 which is required solution.

PROBLEM 2(solution by q method with single parameter)

Basic Analogue of Newton Raphson Method:

Let us solve same problem $f(x)= xe^x-1$

Let $x_0=1$

f(1)= (e-1)

$f(q)=qe^q-1$

Table 4: Calculation of x1 by iterative method (q-analogue of Numerical Methods) for different values of q

q	x_1
0.96	**0.6743415**
0.97	**0.676762956**
0.98	**0.679169779**
0.99	**0.681562017**
0.999999999999	**0.683945343**
1.01	**0.686302938**
1.09	**0.704693177**
0.95	**0.671905363**
0.9	**0.659502787**
0.8	**0.633569653**
0.7	**0.606095896**
0.6	**0.577041065**
0.5	**0.546369238**
0.4	**0.514049563**
0.3	**0.480056757**
0.2	**0.444371563**
0.1	**0.40698115**

Value of x_1 by Newton Raphson Method is 0.6839397.

$f(x_1)$ by Newton Raphson Method is 0.3553424

$f(x_1)$ by q analogue ot Newton Raphson Method at q=0.99 is 0.347423143

$f(x_1)$ by q method at q=0.97 is 0.33153

Observation

We can observe that for q tending to one we get more accurate value of x_1 for which $f(x_1)$ is closer to zero in comparison with classical method. $f(x_1)$ at q=0.99 is 0.347423143 and $f(x_1)$ by Newton Raphson method is 0.3553424. It is apparent that $f(x_1)$ by q-analogue of Newton Raphson method is closer to zero in comparison with $f(x_1)$ calculated by Newton Raphson Method method

.

PROBLEM 3(Example of q-analogue of generalised Newton Raphson Method and q analogue of Newton Raphson Method)

Let us take $f(x)=x^3-x^2-x+1=0$

$x_0=0.8$

Table 5: Calculation of x_1 by iterative method (q-analogue of Numerical Methods) for different values of q

q	$f(qx_0)$	x_1	$f(x_1)$	$f(qx_1)$	x_2
0.9999999	0.072000054	0.905882416	0.016882535	0.016882566	0.954132547
0.999999	0.072000544	0.905882979	0.016882338	0.016882655	0.954132621
0.999	0.072544895	0.906508345	0.016664198	0.016980843	0.954215412
0.99	0.077529088	0.912176313	0.014748616	0.017885707	0.955061074
0.98	0.083234304	0.918543068	0.012729978	0.018936334	0.9562239
0.97	0.089112576	0.924978368	0.01083425	0.020038296	0.957642638
1.000001	0.071999456	0.905881727	0.016882775	0.016882458	0.954132457
0.9999999	0.072000054	0.905882416	0.016882535	0.016882566	0.954132547

Since Exact root is 1

$|\epsilon_0|=|\xi-x_0|=0.8$

$|\epsilon_1|=|\xi-x_1|=0.94118$

$|\epsilon_2|=|\xi-x_2|=0.045868$

$|\epsilon_3|=|\xi-x_3|=0.22662$

If we consider multiplicity which is 2 here then we will use Generalized Newton Raphson Method.

Table 6: Calculation of $x_1$and x_2 by iterative method (q-analogue of Newton Raphson Method with multiplicity) for different values of q

q	x_1	x_2
0.9999999	1.011765	1.000034249
0.999	1.013017	1.000033798
0.99	1.024353	0.9935928
0.98	1.037086	0.985729222
0.97	1.049957	0.977176327
1.000001	1.011763	1.000034801

$|\epsilon_0|=|\xi-x_0|=0.2$

$|\epsilon_1|=|\xi-x_1|=0.12$ E -01

$|\epsilon_2|=|\xi-x_2|=0.34$ E-04

$|\epsilon_3|=|\xi-x_2|=0.74$ E-07

III. Conclusion

q-analogue of iterative methods for solving algebraic and transcendental equations gives the same result as classical methods do but it converges more rapidly towards solution and errors associated with these methods are comparatively lesser if value of q is chosen accordingly and this method is very appropriate for solving transcendental equations .By using single parameter we have to choose value of q very close to one but for double parameter we can get accurate result for most of the values of q_1.Problems have been solved using C++ Programming Language .

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