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Generalized k- Derivations on Lie Ideals of Prime Γ-Rings

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Abstract: *Let M be a 2-torsion free prime Γ- ring and U a Lie ideal of M. Let F : M → M be a mapping defined by F (uαv) = F(u)αv + uk(α)v + uαd(v), for all u ,v ∈ U and α ∈ Γ. Then F is a generalized k-derivation on U of M if there exists a k- derivation d on U of M. Also F is a Jordan generalized k- derivation on U of M if there exists a k - derivation d on U of M such that F(uαu) = F(u)αu + uk(α)u + uαd(u), for all u ∈ U and α ∈ Γ. In this article, we prove that every Jordan generalized k - derivation on a Lie ideal U of a 2 - torsion free prime Γ- ring M is a generalized k - derivation on U of M.*

Keywords: *Lie ideal, k- derivation, generalized k- derivation; Jordan generalized k- derivation, Prime Γ-ring.*

I. Introduction

The Γ- ring is a generalized form of a ring. Nobusawa [1] and Barnes [2] developed the concept of a Γ-ring. The definition of a Γ- ring is as follows :

Let M and Γ be two additive abelian groups. If there is a mapping M × Γ × M → M defined by (x,α, y) → xαy such that ;

(a) (x + y)αz = xαz + yαz , x(α+β)y = xαy + xβy , xα(y + z) = xαy + xαz and

(b) (xαy)βz = xα(yβz) are satisfied for all x, y, z ∈ M and α ∈ Γ, then M is called a Γ- ring in the sense of Bernes [2]. Throughout the paper, we use M as a Γ- ring.

In addition to the definition given above, if there is a mapping Γ × M × Γ → Γ satisfying

(a*) (α + β)xγ = αxγ + βxγ , α(x + y)β = αxβ + αyβ , αx(β + γ) = αxβ + αxγ ;

(b*) (xαy)βz = x(αyβ)z = xα(yβz) ;

(c*) xαy = 0 implies α = 0 , for all x, y, z ∈ M and α, β, γ ∈ Γ; then M is called a Γ- ring in the sense of Nobusawa [1] as simply a Γ_N - ring. It is clear that M is a Γ_N - ring implies that Γ is an M-ring. M is called a prime Γ- ring, if for all x, y ∈ M, xΓMΓy = 0 implies x = 0 or y = 0. And M is called a semiprime Γ- ring if for all x ∈ M, xΓM Γx = 0 implies x = 0. It is clear that every prime Γ- ring is also semi prime but the converse is not true in general. Also M is called a 2- torsion free if 2x = 0 implies x = 0 for every x ∈ M.

The concept of derivations and Jordan derivations were introduced by M. Sapanci and A. Nakajima [3]. H. Kandamar [4] has developed the k-derivation of a Γ- ring. The notion of Jordan k- derivation of a Γ- ring was first introduced by S. Chakraborty and A. C. Paul [5] and proved that every Jordan k- derivation on a 2- torsion free prime Γ_N - ring M is a k- derivation on M. The generalized derivations of a Γ- ring was introduced by Y. Ceven and M. A. Ozturk [6] and proved that every Jordan generalized derivation of a Γ- ring M is a generalized derivation of M. M. M. Rahman and A. C. Paul [7] extended the results of [6] on Lie ideals of prime Γ- rings. In [8], S. Uddin and Paul worked on simple Γ-rings with involutions and extended various results of Herstein [9] in Γ- rings. S. Chakraborty and A. C. Paul [10,11,12,13,14,15] worked on Jordan generalized k-derivations on prime Γ_N - rings, completely prime and completely semiprime Γ_N - rings and developed the various significant results on these fields. The definition of a k- derivation and a Jordan k- derivation are as follows:

Let M be a Γ- ring. Let d: M → M and k: Γ → Γ be an additive mappings. If d(xαy) = d(x)αy + xk(α)y + xαd(y) is satisfied for all x, y ∈ M and α ∈ Γ, then d is said to be a k- derivation of M.

Example : Let R be an associative Ring. Define $M = M_{1,2}(R)$ and $\Gamma = \mathrm{M}_{2,1}(R)$. Then M is a Γ- ring.

Define d : M → M by d((a, b)) = (0, b) and k : Γ→ Γ by $k\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) = \begin{pmatrix}0\\ -\beta\end{pmatrix}$.

Then d is a k- derivation of M for,

$$(0,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y)+(a,b)\begin{pmatrix}0\\ -\beta\end{pmatrix}(x,y)+(a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(0,y)$$

$= (b\beta x, b\beta y) + (-b\beta x, -b\beta y) + (0, a\alpha y + b\beta y)$
$= (b\beta x - b\beta x + 0, b\beta y - b\beta y + a\alpha y + b\beta y)$
$= (0, a\alpha y + b\beta y)$.

Also $(a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y) = (a\alpha x + b\beta x, a\alpha y + b\beta y)$

$\Rightarrow d((a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y)) = d((a\alpha x + b\beta x, a\alpha y + b\beta y))$
$= (0, a\alpha y + b\beta y)$
$= d((a,b))\begin{pmatrix}\alpha\\ \beta\end{pmatrix}(x,y) + (a,b)k\left(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right)(x,y) + (a,b)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}d((0,y))$

And if d(xαx) = d(x)αx + xk(α)x + xαd(x) holds for every x ∈ M and α ∈ Γ, then d is said to be Jordan k-derivation of M. Note that every k-derivation is a Jordan k- derivation but the converse is not true always. Here the notation $[x,y]_\alpha$ is used for the commutator x and y with respect to α, which is defined by $[x,y]_\alpha = x\alpha y - y\alpha x.$ If A is a subset of M , the centre of A with respect to M is Z(A) and is defined by $Z(A) = \{x \in M : [x,a]_\alpha = 0 \text{ for all } a \in A \text{ and } \alpha \in \Gamma\}.$ The centre of a Γ- ring M is denoted by Z(M) and is defined by $Z(M) = \{x \in M : [x,y]_\alpha = 0 \text{ for all } y \in M \text{ and } \alpha \in \Gamma\}.$ A Γ- ring M is commutative if and only if M = Z(M).

Throughout this paper, we shall use the condition (*) aαbβc = aβbαc for all a, b, c ∈ M and α , β ∈ Γ. By the condition , the commutator identities
$[a\alpha b, x]_\beta = [a,x]_\beta \alpha b + a[\alpha,\beta]_x b + a\alpha[b,x]_\beta$ and $[x, a\alpha b]_\beta = a\alpha[x,b]_\beta + a[\alpha,\beta]_\beta b + [x,a]_\beta \alpha\beta$
given in [4] reduces to $[a\alpha b, x]_\beta = a\alpha[b,x]_\beta + [a,x]_\beta \alpha b$ and $[x, a\alpha b]_\beta = a\alpha[x,b]_\beta + [x,a]_\beta \alpha b.$

In this paper, we prove that every Jordan generalized k- derivation on a Lie ideal U of M is a generalized k-derivation on U of M.

II. Generalized and Jordan Generalized k- Derivation

Definition 2.1. Let M be a Γ-ring and let k : Γ → Γ be an additive mapping. An additive mapping F: M → M is called a generalized k- derivation if there exists a k- derivation d : M → M such that F(uαv) = F(u)αv + uk(α)v + uαd(v) for all u ,v ∈ M and α ∈ Γ. And if F(uαu) = F(u)αu + uk(α)u + uαd(u) holds for all u ∈ M and α ∈ Γ, then F is said to be a Jordan generalized k- derivation.

Example : Let M be a Γ- ring and let F be a generalized k- derivation of M.Then by definition, there exists a k-derivation d: M→M such that d(xαy) = d(x)αy + xk(α)y + xαd(y) and F(xαy) = F(x) αy + xk(α)y + xαd(y) , for all x, y ∈ M and α ∈ Γ.

Let $M_1 = M \times M$ and $\Gamma_1 = \Gamma \times \Gamma$. Define the operations of addition and multiplication of M_1 and Γ_1 by (x, y) + (z, w) = (x + z, y + w) and (x, y)(α, β)(z, w) = (xαz, yβw) for every x, y, z, w ∈ M and α, β ∈ Γ. Then M_1 is obviously a Γ_1 - ring under these operations.

Let $F_1 : M_1 \to M_1, d_1 : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ be the additive mappings defined by $F_1((x,y)) = (F(x), F(y)),\quad d_1((x,y)) = (d(x), d(y))$ and $k_1((\alpha,\beta)) = (k(\alpha), k(\beta))$ for all x, y ∈ M and α ,β ∈ Γ. Then clearly d_1 is a k_1- derivation of M_1.

Put $(x,y) = a \in M_1, (\alpha,\beta) = \gamma \in \Gamma_1$, for any x, y ∈ M and α ,β ∈ Γ ; then we have,

$F_1(a\gamma a) = F_1((x,y)(\alpha,\beta)(x,y))$
$= F_1(x\alpha x, y\beta y)$
$= (F(x\alpha x), F(y\beta y))$
$= (F(x)\alpha x + xk(\alpha)x + x\alpha d(x), F(y)\beta y + yk(\beta)y + y\beta d(y))$
$= (F(x)\alpha x, F(y)\beta y) + (xk(\alpha)x, yk(\beta)y) + (x\alpha d(x), y\beta d(y))$
$= (F(x), F(y))(\alpha,\beta)(x,y) + (x,y)(k(\alpha),k(\beta))(x,y) + (x,y)(\alpha,\beta)(d(x),d(y))$
$= F_1(x,y)(\alpha,\beta)(x,y) + (x,y)k_1(\alpha,\beta)(x,y) + (x,y)(\alpha,\beta)d_1(x,y)$
$= F_1(a)\gamma a + ak_1(\gamma)a + a\gamma d_1(a),$

which follows that F_1 is a Jordan generalized k_1-derivation of M_1 associated with the k_1-derivation d_1 of M_1.

Definition 2.2. Let M be a Γ- ring. An additive subgroup U of M is called a Lie ideal of M if $[u,m]_\alpha \in U$ for every u ∈ U , m ∈ M and α ∈ Γ. Note that every ideal of a Γ- ring M is a Lie ideal of M but the converse is not true in general.

Example: Let R be a ring and U be a Lie ideal of R. Let $M = M_{1,2}(\mathrm{R})$ and $\Gamma = \left\{\begin{pmatrix} n.1 \\ 0 \end{pmatrix} : n \in Z\right\}$. Then M is a Γ- ring . Define N = {(x , x) : x ϵ R } ⊆ M . Then N is a Γ- ring. Let $U_1 = \{(u,u) : u \in U\}$.

Now $(u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) - (a,a)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)$
$= (una, una) - (anu, anu)$
$= (una - anu, una - anu) \in U_1.$

Then U is a Lie ideal of N. It is clear that U_1 is not an ideal of N.

Definition 2.3. Let M be a Γ- ring and let U be a Lie ideal of M. Let k : Γ → Γ be an additive mapping. An additive mapping F: M → M is called a generalized k- derivation on U of M if there exists a k- derivation d on U of M such that F (uαv) = F(u)αv + uk(α)v + uαd(v) for all u ,v ∈ U and α ∈ Γ.

Example: Let M be a Γ - ring and let U be a Lie ideal of M. Let f: M → M be a generalized k- derivation on U of M, then there exists a derivation d on U of M such that f(uαv) = f(u)αv + uk(α)v + uαd(v) for all u, v ∈ U and α ∈ Γ. Let $M_1 = \{(x,x) : x \in M\}$ and $\Gamma_1\{(\alpha,\alpha) : \alpha \in \Gamma\}$. Define addition and multiplication of M are as follows:

(x, x) + (y, y) = (x + y, x + y) , (x, x)(α, α)(y, y) = (xαy, xαy) for all $(x,x) \in M_1$ and $(\alpha,\alpha) \in \Gamma_1$. Under these operations M_1 is a Γ_1 - ring.

Let $U_1 = \{(u,u) : u \in U\}$. Then clearly U_1 is a Lie ideal of M_1. Define $F : M_1 \to M_1, D : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ by $F((x,x)) = (f(x),f(x)), D((x,x)) = (d(x),d(x))$ and $\mathrm{k}_1((\alpha,\alpha)) = (k(\alpha),k(\alpha))$ for all $x \in U$ *and* $\alpha \in \Gamma$.

Then $F((x,x)(\alpha,\alpha)(y,y)) = F(x\alpha y, x\alpha y))$
$= (f(x\alpha y), f(x\alpha y))$
$= (f(x)\alpha y + xk(\alpha)y + x\alpha d(y), f(x)\alpha y + xk(\alpha)y + x\alpha d(y))$
$= (f(x)\alpha y, f(x)\alpha y)) + (xk(\alpha)y, xk(\alpha)y) + (x\alpha d(y), x\alpha d(y))$

$$= (f(x), f(x))(\alpha,\alpha)(y,y) + (x,x)(k(\alpha),k(\alpha))(y,y) + (x,x)(\alpha,\alpha)(d(y),d(y))$$
$$= F((x,x))(\alpha,\alpha)(y,y) + (x,x)k_1((\alpha,\alpha))(y,y) + (x,x)(\alpha,\alpha)D((y,y))$$

Therefore F is a generalized k- derivation on U_1 of M_1.

Also F : M → M is called a Jordan generalized k- derivation on U of M if there exist a k- derivation d on U of M such that F(uαu) = F(u)αu + uk(α)u + uαd(u) , for every u ∈ U and α ∈ Γ.

Example: Let M be a Γ - ring and let U be a Lie ideal of M. Let f: M → M be a generalized k- derivation on U of M, then there exists a derivation d on U of M such that f(uαv) = f(u)αv + uk(α)v + uαd(v) for all u, v ∈ U and α ∈ Γ.
Let $M_1 = \{(x,x) : x \in M\}$ and $\Gamma_1 : \{(\alpha,\alpha) : \alpha \in \Gamma\}$. Define addition and multiplication of M are as follows:
(x, x) + (y, y) = (x + y, x + y) , (x, x)(α, α)(y, y) = (xαy, xαy) for all $(x,x) \in M_1$ and $(\alpha,\alpha) \in \Gamma_1$. Under these operations M_1 is a Γ_1 - ring. Let $U_1 = \{(u,u) : u \in U\}$. Then clearly U_1 is a Lie ideal of M_1. Define
$F : M_1 \to M_1, D : M_1 \to M_1$ and $k_1 : \Gamma_1 \to \Gamma_1$ by $F((x,x)) = (f(x), f(x)), D((x,x) = (d(x), d(x))$ and $k_1((\alpha,\alpha)) = (k(\alpha), k(\alpha))$ for all $x \in U$ and $\alpha \in \Gamma$.
Then $F((x,x)(\alpha,\alpha)(x,x)) = F((x\alpha x, x\alpha x))$
$$= (f(x\alpha x), f(x\alpha x))$$
$$= (f(x)\alpha x + xk(\alpha)x + x\alpha d(x), f(x)\alpha x + xk(\alpha)x + x\alpha d(x))$$
$$= (f(x)\alpha x, f(x)\alpha x) + (xk(\alpha)x, xk(\alpha)x) + (x\alpha d(x), x\alpha d(x))$$
$$= (f(x), f(x))(\alpha,\alpha)(x,x) + (x,x)(k(\alpha),k(\alpha))(x,x) + (x,x)(\alpha,\alpha)(d(x),d(x))$$
$$= F((x,x))(\alpha,\alpha)(x,x) + (x,x)k_1((\alpha,\alpha))(x,x) + (x,x)(\alpha,\alpha)D((x,x)).$$
Therefore F is a Jordan generalized k- derivation on U_1 of M_1.

Lemma 2.4 Let M be a 2- torsion free Γ -ring satisfying (*) and U a Lie ideal of M such that uαu ∈ U for all u ∈ U and . Let F : M → M be a Jordan generalized k- derivation on U, then

(i) F(uαv +vαu) = F(u)αv +uk(α)v + uαd(v) +F(v)αu + vk(α)u + vαd(u).
(ii) F(uαvβu) = F(u)αvβu + uk(α)vβu + uαd(v)βu + uαvk(β)u + uαvβd(u)
(iii) F(uαvβw + wαvβu) = F(u)αvβw + uk(α)vβw +uαd(v)βw +uαvk(β)w + uαvβd(w) + F(w)αvβu + wk(α)vβu + wαd(v)βu + wαvk(β)u + wαvβd(u).

Proof: We have uαv + vαu = (u + v)α(u + v) - uαu - vαv, and the left side as like as the right side is in U.
Hence F(uαv +vαu) = F((u+v)α (u+v) - uαu - vαv)
= F (u+v)α(u+v) + (u+v)k(α)(u + v) + (u+v)αd(u+v) -(F(u)αu + uk(α)u + uαd(u) + F(v)αv + vk(α)v + vαd(v))
= F(u)αu + F(u)αv +F(v)αu +F(v) αv + uk(α)u + uk(α)v + vk(α)u + vk(α)v + uαd(u)+ uαd(v) +vαd(u) +vαd(v) - F(u)αu - uk(α)u - uαd(u) -F(v)αv - vk(α)v -vαd(v).
⇒ F(uαv + vαu) = F(u)αv + uk(α)v + uαd(v) + F(v)α u + vk(α)u + vαd(u).
Replacing v by uβv + vβu we have,
F(uα(uβv +vβu) +(uβv +vβu)αu) = F(u)α(uβv+vβu) +uk(α) (uβv +vβu) + uαd(uβv + vβu) + F(uβv + vβu)αu + (uβv +vβu) k(α)u + (uβv + vβu)α d(u)…………….(1)
Left side of (1) is equal to
F(uαuβv + uαvβu + uβvαu + vβuαu) = F(uαvβu + uβvαu) + F((uαu)βv + vβ(uαu))
=F(uαvβu + uβvαu) + F(uαu)βv + uαuk(β)v + uαuβd(v) + F(v)βuαu + vk(β)uαu + vβd(uαu)
= F(uαvβu + uβvαu) + F(u)αuβv + uk(α)uβv + uαd(u)βv + uαuk(β)v + uαuβd(v) + F(v)βuαu + vk(β)uαu + vβd(u)αu + vβuk(α)u + vβuαd(u).
Right side of (1) is equal to
F(u)αuβv + F(u)αvβu + uk(α)uβv + uk(α)vβu + uαd(u)βv + uαuk(β)v + uαuβd(v) +uαd(v)βu + uαvk((β)u + uαvβd(u)+F(u)βvαu+ uk(β)vαu + uβd(v)αu +F(v)βuαu + vk(β)uαu + vβd(u)αu + uβvk(α)u + vβuk(α)u + uβvαd(u) + vβuαd(u).

Computing both sides we have,
$F(u\alpha v\beta u + u\beta v\alpha u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(u)\beta v\alpha u + uk(\beta)v\alpha u + u\, d(v)\alpha u + u\beta vk(\alpha)u + u\beta v\alpha d(u)$.
Putting $u\beta v\alpha u = u\alpha v\beta u$ we have ,
$F(2u\alpha v\beta u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(u)\alpha v\beta u + u\alpha vk(\beta)u + u\alpha d(v)\beta u + uk(\alpha)v\beta u + u\alpha v\beta d(u)$
$\Rightarrow 2F(u\alpha v\beta u) = 2(F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha\, d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u))$.
Since M is a 2- torsion free, hence we have
$F(u\alpha v\beta u) = F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u)$.
Replace $u + w$ for u we have,
$F((u+w)\alpha v\beta(u+w)) = F(u+w)\alpha v\beta(u+w)+ (u+w)k(\alpha)v\beta(u+w) + (u+w)\alpha d(v)\beta(u+w) +(u+w)\alpha vk(\beta)(u+w) + (u+w)\alpha v\beta d(u+w)$ …………….. (2)
Left side of (2) is equal to
$F(u\alpha v\beta u + u\alpha v\beta w + w\alpha v\beta u + w\alpha v\beta w) = F(u\alpha v\beta w +w\alpha v\beta u) + F(u\alpha v\beta u) + F(w\alpha v\beta w)$
$= F(u\alpha v\beta w + w\alpha v\beta u) + F(u)\alpha v\beta u + uk(\alpha)v\beta u + u\alpha d(v)\beta u + u\alpha vk(\beta)u + u\alpha v\beta d(u) + F(w)\alpha v\beta w + wk(\alpha)v\beta w + w\alpha d(v)\beta w + w\alpha vk(\beta)w+ w\alpha v\beta d(w)$.
Right side of (2) is equal to
$F(u)\alpha v\beta u + F(w)\alpha v\beta u + F(u)\alpha v\beta w + F(w)\alpha v\beta w + uk(\alpha)v\beta u + wk(\alpha)v\beta u + uk(\alpha)v\beta w + wk(\alpha)v\beta w + u\alpha d(v)\beta u +w\alpha d(v)\beta u + u\alpha d(v)\beta w + w\alpha d(v)\beta w + u\alpha vk(\beta)u + w\alpha vk(\beta)u + u\alpha vk(\beta)w + w\alpha vk(\beta)w + u\alpha v\beta d(u) + u\alpha v\beta d(w) + w\alpha v\beta d(u) + w\alpha v\beta d(w)$.
Comparing both sides we get,
$F(u\alpha v\beta w + w\alpha v\beta u) = F(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + F(w)\alpha v\beta u + wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)$.

Definition: We define $\psi_\alpha(u,v) = F(u\alpha v) - F(u)v - uk(\alpha)v - u\alpha d(v)$ for all $u,v \in U$ and $\alpha \in \Gamma$.

Remark 2.6 : It is clear that F is a generalized k- derivation if and only if $\Psi_\alpha(u,v) = 0$.

Lemma 2.7 : Let M , U and F be as in above. Then for all u, v, w $\in$ U and $\alpha, \beta \in \Gamma$, the following relations hold :

$(i)\quad \Psi_\alpha(u,v) + \Psi_\alpha(v,u) = 0$

$(ii)\quad \Psi_\alpha(u+w,v) = \Psi_\alpha(u,v) + \Psi_\alpha(w,v)$

$(iii)\quad \Psi(u,v+w) = \Psi_\alpha(u,v) + \Psi_\alpha(u,w)$

$(iv)\quad \Psi_{\alpha+\beta}(u,v) = \Psi_\alpha(u,v) + \Psi_\beta(u,v)$.

Lemma 2.8 : Let M , U , F and d be defined as in above , then for all u, v , w $\in$ U and $\alpha, \beta, \gamma \in \Gamma$,
$\Psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0$.

Proof: Consider $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v)$.
From Lemma 2.4 (iii) we have, $F(A) = F((2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v))$

$$= F(2u\alpha v)\beta w\gamma(2v\alpha u) + 2u\alpha vk(\beta)w\gamma(2v\alpha u) + 2u\alpha v\beta d(w)\gamma(2v\alpha u) + 2u\alpha v\beta wk(\gamma)(2v\alpha u) + (2u\alpha v)\beta w\gamma d(2v\beta u) + F(2v\alpha u)\beta w\gamma(2u\alpha v) + (2v\alpha u)k(\beta)w\gamma(2u\alpha v) + (2v\alpha u)\beta d(w)\gamma(2u\alpha v) + (2v\alpha u)\beta wk(\gamma)(2u\alpha v) + (2v\alpha u)\beta w\gamma d(2u\alpha v)$$

$$= 4[F(u\alpha v)\beta w\gamma(v\alpha u) + u\alpha vk(\beta)w\gamma v\alpha u + u\alpha v\beta d(w)\gamma v\alpha u + u\alpha v\beta wk(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v\beta u) + F(v\alpha u)\beta w\gamma u\alpha v + v\alpha uk(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta wk(\gamma)u\alpha v + v\beta u\beta w\gamma d(u\alpha v)].$$

Again $A = (2u\alpha v)\beta w\gamma(2v\alpha u) + (2v\alpha u)\beta w\gamma(2u\alpha v) = u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v$

$\Rightarrow F(A) = F(u\alpha(4v\beta w\gamma v)\alpha u + v\alpha(4u\beta w\gamma u)\alpha v)$
$= 4[F(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v\beta w\gamma v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) +$
$F(v)\alpha u\beta w\gamma u\alpha v + vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u\beta w\gamma u)\alpha v + v\alpha u\beta w\gamma uk(\alpha)v + v\alpha u\beta w\gamma u\alpha d(v)]$ using lemma 2.4(ii)
$= 4[F(u)\alpha v\beta w\gamma v\alpha u + uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha vk(\beta)w\gamma v\alpha u + u\alpha v\beta d(w)\gamma v\alpha u +$
$u\alpha v\beta wk(\gamma)v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + F(v)\alpha u\beta w\gamma u\alpha v +$
$vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha uk(\beta)w\gamma u\alpha v + v\alpha u\beta d(w)\gamma u\alpha v + v\alpha u\beta wk(\gamma)u\alpha v +$
$v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma uk(\alpha)v + v\alpha u\beta w\gamma u\alpha d(v)]$

Comparing both expressions we have,

$$4[F(u\alpha v)\beta w\gamma v\alpha u + F(v\alpha u)\beta w\gamma u\alpha v + u\alpha v\beta w\gamma d(v\alpha u) + v\alpha u\beta w\gamma d(u\alpha v)] = 4[F(u)\alpha v\beta w\gamma v\alpha u +$$
$$uk(\alpha)v\beta w\gamma v\alpha u + u\alpha d(v)\beta w\gamma v\alpha u + u\alpha v\beta w\gamma d(v)\alpha u + u\alpha v\beta w\gamma vk(\alpha)u + u\alpha v\beta w\gamma v\alpha d(u) + F(v)\alpha u\beta w\gamma u\alpha v +$$
$$vk(\alpha)u\beta w\gamma u\alpha v + v\alpha d(u)\beta w\gamma u\alpha v + v\alpha u\beta w\gamma d(u)\alpha v + v\alpha u\beta w\gamma uk(\alpha)v + v\alpha u\beta w\gamma u\alpha d(v)]$$

Since M is a 2- torsion free , we have

$$[F(u\alpha v) - F(u)\alpha v - uk(\alpha)v - u\alpha d(v)]\beta w\gamma v\alpha u + [F(v\alpha u) - F(v)\alpha u - vk(\alpha)u - v\alpha d(u)]\beta w\gamma u\alpha v +$$
$$u\alpha v\beta w\gamma[d(v\alpha u) - d(v)\alpha u - vk(\alpha)u - v\alpha d(u)] + v\alpha u\beta w\gamma[d(u\alpha v) - d(u)\alpha v - uk(\alpha)v - u\alpha d(v)] = 0$$

$$\Rightarrow \psi_\alpha(u,v)\beta w\gamma v\alpha u + \psi_\alpha(v,u)\beta w\gamma u\alpha v + [d(v\alpha u) - d(v\alpha u)] + [d(u\alpha v) - d(u\alpha v)] = 0$$
$$\Rightarrow \psi_\alpha(u,v)\beta w\gamma v\alpha u - \psi_\alpha(u,v)\beta w\gamma u\alpha v = 0$$
$$\Rightarrow -\psi_\alpha(u,v)\beta w\gamma(v\alpha u - u\alpha v) = 0$$
$$\Rightarrow \psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha = 0$$

Lemma 2.9: [16, Lemma 2.10] Let U ⊄ Z(M) be a Lie ideal of a Prime Γ- ring M satisfying the condition (*) and a, b ∈ M (res. b ∈ U and a ∈ M) such that aαUβb = 0 for all α , β ∈ Γ, then a = 0, or b = 0.

Lemma 2.10 : Let U ⊄ Z(M) be a Lie ideal of a 2- torsion free prime Γ- ring M. Then $[u,v]_\alpha \beta w\gamma\psi_\alpha(u,v) = 0$.

Proof: From Lemma 2.8 we have, $\psi_\alpha(u,v)\delta x\mu[u,v]_\alpha = 0$
$\Rightarrow [u,v]_\alpha \beta w\gamma\psi_\alpha(u,v)\delta x\mu[u,v]_\alpha \beta w\gamma\psi_\alpha(u,v) = 0$ for all $x \in U$.
In view of Lemma 2.9, we have $[u,v]_\alpha \beta w\gamma\psi_\alpha(u,v) = 0$.

Lemma 2.11: Let U ⊄ Z(M) be a Lie ideal of a 2- torsion free prime Γ-ring . Then $\psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0$ for all u, v , w, x , y ∈ U and α, β, γ, δ ∈ Γ.

Proof: From Lemma 2.8 we have $\psi_\alpha(u+x,v)\beta w\gamma[u+x,v]_\alpha = 0$
$$\Rightarrow \psi_\alpha(u,v)\beta w\gamma[u,v]_\alpha + \psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha + \psi_\alpha(x,v)\beta w\gamma[u,v]_\alpha + \psi_\alpha(x,v)\beta w\gamma[x,v]_\alpha = 0$$
$$\Rightarrow \Psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha + \Psi_\alpha(x,v)\beta w\gamma[u,v]_\alpha = 0$$
$$\Rightarrow \psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha = -\psi_\alpha(x,v)\beta w\gamma[u,v]_\alpha$$
$$\Rightarrow (\psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha)\partial p\theta(\psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha)$$
$$= -\psi_\alpha(x,v)\beta w\gamma[u,v]_\alpha \delta p\theta\psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha$$
$$= -\psi_\alpha(x,v)\beta w\gamma([u,v]_\alpha \delta p\theta\psi_\alpha(u,v))\beta w\gamma[x,v]_\alpha = 0 \quad \text{[by lemma 2.8]}$$

By Lemma 2.9, we have $\psi_\alpha(u,v)\beta w\gamma[x,v]_\alpha = 0$

Likewise by replacing v + y for v we get $\psi_\alpha(u,v)\beta w\gamma[x,y]_\alpha = 0.....................(i)$

Proceeding in the same way as above , by the similar replacement in the result, we have

$[x,y]_\alpha \beta w\gamma\psi_\alpha(u,v) = 0$

Now putting α + δ for α in (i) we have, $\psi_{\alpha+\delta}(u,v)\beta w\gamma[x,y]_{\alpha+\delta} = 0$

$\Rightarrow \psi_\alpha(u,v)\beta w\gamma[x,y]_\alpha + \psi_\delta(u,v)\beta w\gamma[x,y]_\alpha + \psi_\alpha(u,v)\beta w\gamma[x,y]_\delta + \psi_\delta(u,v)\beta w\gamma[x,y]_\delta = 0$

$\Rightarrow \psi_\delta(u,v)\beta w\gamma[x,y]_\alpha + \psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0$

$\Rightarrow \psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = \psi_\delta(u,v)\beta w\gamma[x,y]_\alpha$

Therfore,

$\psi_\alpha(u,v)\beta w\gamma[x,y]_\delta \theta q\mu\psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = -\psi_\delta(u,v)\beta w\gamma([x,y]_\alpha \theta q\mu\psi_\alpha(u,v))\beta w\gamma[x,y]_\delta = 0$ by (i)

Using Lemma 2.9, we have $\psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0.$

Lemma 2.12: Let U ⊄ Z(M) be a Lie ideal of a 2- torsion free prime Γ- ring M , then Z(U) = Z(M).

Proof: We have Z(U) is both a sub Γ-ring and a Lie ideal of M . Also we know that Z(U) cannot contain a nonzero ideal of M. So by [17, Lemma 3.7], Z(U) is contained in Z(M) . Therefore , Z(U) = Z(M).

Lemma 2.13: Let U ⊄ Z(M) be a Lie ideal of a 2- torsion free prime Γ- ring M . Then $\psi_\alpha(u,v) \in Z(U) = Z(M)$ for every $u,v \in U, \alpha \in \Gamma$..

Proof : We have $\psi_\alpha(u,v)\beta w\gamma([x,y]_\delta) = 0$

Now $2[\psi_\alpha(u,v),c]_\delta \beta w\gamma[\psi_\alpha(u,v),c]_\delta$

$= 2\psi_\alpha(u,v)\delta c - c\delta\psi_\alpha(u,v))\beta w\gamma[\psi_\alpha(u,v),c]_\delta$

$= \psi_\alpha(\mathrm{u,v})\delta(2c\beta w)\gamma[\psi_\alpha(u,v),c]_\delta - 2c\delta\psi_\alpha(u,v)\beta w\gamma[\psi_\alpha(u,v),c]_\delta = 0,$ for every $c \in U.$

In view of Lemma 2.9, we have $[\Psi_\alpha(u,v),c]_\delta = 0$,

$\Rightarrow \psi_\alpha(u,v) \in Z(U)$ and that implies $\psi_\alpha(u,v) \in Z(M)$ by Lemma 2.12.

Lemma 2.14: Let M be a 2- torsion free prime Γ- ring satisfying the condition (*) and U a Lie ideal of M . Let $u \in U$ be such that $[u.[u,x]_\alpha]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$. Then $[u,x]_\alpha = 0$

Proof: We have $[u,[u,x]_\alpha]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$.

Let y ∈ M , then xαy ∈ M for all α ∈ Γ . Replace x by xαy we have $[u,[u,x\alpha y]_\alpha]_\alpha = 0$

$\Rightarrow [u,(x\alpha[u,y]_\alpha + [u,x]_\alpha)\alpha y]_\alpha = 0$

$\Rightarrow [u,x\alpha[u,y]_\alpha]_\alpha + [u,[u,x]_\alpha \alpha y]_\alpha = 0$

$\Rightarrow x\alpha[u,[u,y]_\alpha]_\alpha + [u,x]_\alpha \alpha[u,y]_\alpha + [u,x]_\alpha \alpha[u,y]_\alpha + [u,[u,x]_\alpha]_\alpha \alpha y = 0$

$\Rightarrow 2[u,x]_\alpha \alpha[u,y]_\alpha = 0$

Since M is 2 - torsion free , we have $[u,x]_\alpha \alpha[u,y]_\alpha = 0$.

Putting y = uβx , we have $[u,x]_\alpha \alpha[u,u\beta x]_\alpha = 0 \Rightarrow [u,x]_\alpha \alpha u\beta[u,x]_\alpha = 0$ by using (*) .

Hence by Lemma 2.9 we have $[u,x]_\alpha = 0$.

Lemma 2.15 : Let M be a 2-torsion free prime Γ- ring satisfying the condition (*) and U be a commutative Lie ideal of M , then U ⊆ Z(M).

Proof : Since U is commutative ,we have $[u,v]_\alpha = 0$ for all $u,v \in U$ and $\alpha \in \Gamma.$

Also we have $[u,x]_\alpha \in U$ for all $u \in U, x \in M$ and $\alpha \in \Gamma.$

Replacing v by $[u,x]_\alpha$, we obtain $[u,[u,x]_\alpha]_\alpha = 0$

By Lemma 2.14 we have $[u,x]_\alpha = 0.$ Hence $U \subseteq Z(M)$.

Theorem 2.16: Let M be a 2- torsion free prime Γ-ring satisfying the condition (*) and let U be a Lie ideal of M such that uαu ∈ U for all u ∈ U and if F : M →M is a Jordan generalized derivation on U of M then, $\psi_\alpha(u,v) = 0$ for all u , v ∈ U and α ∈ Γ.

Proof: Let U be a commutative Lie ideal of M . Then by Lemma 2.14, $U \subseteq Z(M)$.

Since U is commutative, then $[v,w]_\beta = 0$ implies $v\alpha w = w\alpha v,$, for every $v,w \in U, \alpha \in \Gamma$.

From Lemma 2.4 (iii) we have,

$$F(u\alpha v\beta w + w\alpha v\beta u) = F(u)\alpha v\beta w + uk(\alpha)v\beta w + u\alpha d(v)\beta w + u\alpha vk(\beta)w + u\alpha v\beta d(w) + F(w)\alpha v\beta u +$$
$$wk(\alpha)v\beta u + w\alpha d(v)\beta u + w\alpha vk(\beta)u + w\alpha v\beta d(u)..................(i)$$

Putting u = 2vβw in (1), we have

$$\text{L.S.} = F(2v\beta w\alpha v\beta w + w\alpha v\beta 2v\beta w)$$
$$= 2F(v\beta w\alpha v\beta w + w\beta v\alpha v\beta w)$$
$$= 2F(v\beta w\alpha v\beta w + v\beta w\alpha v\beta w)$$
$$= 4F((v\beta w)\alpha(v\beta w))$$
$$= 4(F(v\beta w)\alpha(v\beta w) + v\beta wk(\alpha)v\beta w + v\beta w\alpha d(v\beta w)).$$

$$\text{Also R.S.} = F(2v\beta\beta)\alpha v\beta w + 2v\beta\beta w(\alpha)v\beta w + 2v\beta w\alpha d(v)\beta w + 2v\beta w\alpha vk(\beta)w + 2v\beta w\alpha vd(w) +$$
$$F(w)\alpha v\beta 2v\beta w + wk(\alpha)v\beta 2v\beta w + w\alpha d(v)\beta 2v\beta w + w\alpha vk(\beta)2v\beta w + w\alpha v\beta d(2v\beta w)$$
$$= F(2v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w + 2v\beta w\alpha[d(v)\beta w + vk(\beta)w + v\beta d(w)] + 2F(w)\alpha v\beta v\beta w +$$
$$2wk(\alpha)v\beta v\beta w + 2w\alpha d(v)\beta v\beta w + 2w\alpha vk(\beta)v\beta w + 2w\alpha v\beta d(v\beta w)$$
$$= 2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w + 4v\beta w\alpha d(v\beta w) + 2F(w)\beta v\alpha v\beta w + 2wk(\alpha)v\beta v\beta w + 2w\alpha d(v)\beta v\beta w +$$
$$wk(\beta)v\alpha v\beta w.$$

Comparing both sides we get

$$2F(v\beta w)\alpha v\beta w + 2v\beta wk(\alpha)v\beta w - 2F(w)\beta v\alpha v\beta w - 2wk(\beta)v\alpha v\beta w - 2w\beta d(v)\alpha v\beta w - 2v\beta wk(\alpha)v\beta w = 0$$
$$\Rightarrow 2(F(w\beta v) - F(w)\beta v - wk(\beta)v - w\beta d(v))\alpha v\beta w = 0$$
$$\Rightarrow 2\psi_\beta(w.v)\alpha v\beta w = 0 \Rightarrow \psi_\beta(w,v)\alpha v\beta w = 0$$
$$\Rightarrow \psi_\beta(w,v)\alpha v\beta w\gamma x\delta y = 0, \text{ where } x \in U, y \in M.$$
$$\Rightarrow \psi_\beta(w,v)\alpha x\beta y\gamma v\delta w = 0$$
$$\Rightarrow (\psi_\beta(w,v)\alpha x\gamma y)\beta v\delta w = 0 \quad \text{using } (*).$$

From Lemma 2.9, either $\psi_\beta(w,v)\alpha x\gamma y = 0$ or $w = 0$.

Since $w \in U, w \neq 0,$ hence $\psi_\beta(w,v)\alpha x\gamma y = 0.$ That implies $\psi_\beta(w,v)\alpha U\gamma y = 0.$

Using Lemma 2.9 we have , $\psi_\beta(w,v) = 0.$

Again if U is not commutative , i.e., $U \not\subset Z(M)$, then from Lemma 2.11 we have

$\psi_\alpha(u,v)\beta w\gamma[x,y]_\delta = 0.$ But $[x,y]_\delta = 0$ implies $U \subseteq Z(M),$ a contradiction .

Hence $\psi_\alpha(u,v) = 0,$ for all $u,v \in U$ and $\alpha \in \Gamma.$

Corollary 2.17: Every Jordan generalized k-derivation of a 2 -torsion free prime Γ- ring M is a generalized k-derivation on M.

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