Almost Closed and Continuous Maps

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Abstract: *The primary purpose of this paper is to establish the relation between continuous maps and almost closed maps. In this endeavor some known results are improved.*
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I.   Introduction

By a space, we shall mean a topological space. No separation axioms are assumed and no map is assumed to be continuous or onto unless mentioned explicitly; cl(A) will denote the closure of the subset A in the space X. X is said to be Frechet space (or closure sequential in the terminology of Wilansky [1]) if for each subset A of X, $x \in cl(A)$ implies there exists a sequence $\{x_n\}$ in A converging to x. If X and Y are topological spaces, we say that a map f: $X \longrightarrow Y$ has a closed graph if G(f) = $\{(x,f(x)): x \in X\}$ is a closed subset of X x Y. A map f :$X \longrightarrow Y$ is said to be almost closed if f(V) is closed for every regular closed subset V of X. A space X is said to be almost compact if for every open cover $\{U_\alpha : \alpha \in \Lambda \}$ there exists a finite subfamily $\{ U_{\alpha_i} : i = 1,2,\ldots,n\}$ such that $\bigcup_{i=1}^{n} cl(U_{\alpha_i}) = X$. If f: $X \longrightarrow Y$ is a map , then the fibers are the sets $f^{-1}(y)$, where $y \in Y$. X is said to be B-W compact [2] if every infinite subset of X has at least one limit point .

The primary purpose of this paper is to give the relationship between almost closed maps and continuous maps. The following theorem 2.2 gives the conditions under which an almost closed map becomes continuous while theorem 2.4 gives the conditions under which a continuous map becomes almost closed. Moreover, some established results are improved.
In 1968,Fuller [3] has proved the following

Theorem 1.1[3,Corollary 3.9] . Let f:$X \longrightarrow Y$ be closed with closed fibers where X is regular. Then f has closed graph.
In 1997,Garg and Goel [4] have proved the following

Theorem 1.2 [4,Theorem 2]. Let f:$X \longrightarrow Y$ be closed with compact fibers where X is T_2. Then f has closed graph.

Theorem 1.3[4,Theorem 4]. Let f:$X \longrightarrow Y$ have closed graph, then f is continuous if any one of the following conditions is satisfied.
(a)   Y is compact
(b)   X is Frechet ,Y is B-W compact
(c)   f is subcontinuous

Theorem 1.4[4,Theorem 7]. Let f:$X \longrightarrow Y$ be closed with(closed) compact fibers, where X is (regular)T_2 and one of the conditions (a),(b),(c) in theorem 1.3 is satisfied. Then f is continuous.

II.   Main Results

The following theorem 2.1 (theorem 2.2) is a generalisation of the above theorems 1.1 and theorem 1.2 (theorem 1.4) as the condition of closedness of f is weakened to almost closedness.
Theorem 2.1 . Let f:$X \longrightarrow Y$ be almost closed with(closed) compact fibers, where X is (regular)T_2. Then f has closed graph.
Proof. We prove the parenthesis part , the proof of the non parenthesis part is similar. Let $x \in X, y \in Y, y \neq f(x)$. Then $x \notin f^{-1}(y)$ which is closed. Since X is regular, there exist open sets U_x and U_y containing x and $f^{-1}(y)$ respectively such that $U_x \bigcap U_y = \phi$ which implies $cl(U_x) \bigcap U_y = \phi$. As $f^{-1}(y) \subset U_y$ this implies $f^{-1}(y) \bigcap cl(U_x) = \phi$ and therefore $y \notin f(cl(U_x))$.Now f is almost closed map and $cl(U_x)$ is a almost regular set implies

$f(cl(U_x))$ is a closed set. Hence $V=Y-f(cl(U_x))$ is an open set containing y and $f(U_x)\cap V=\phi$ implying thereby that f has closed graph.

Combining our theorem 1.3 and above theorem 2.1, we obtain the following

Theorem 2.2. Let $f:X\rightarrow Y$ be almost closed with(closed) compact fibers, where X is (regular)T_2 and one of the conditions (a),(b),(c) in theorem 1.3 is satisfied. Then f is continuous.

The following lemma will be used in the proof of our theorem 2.4 which gives the conditions under which the converse of theorem 2.2 holds.

Lemma 2.1. Almost compact subset of a T_2 space is closed.

Proof. Let K be any almost compact subset of T_2 space X and let $x\notin K$ which implies $x\neq y$, $y\in K$. Then X is T_2 implies there exist open sets U_x and U_y containing x and y respectively such that $U_x\cap U_y=\phi$ and therefore $U_x\cap clU_y=\phi$ .Now $\{U_y\}$ $y\in K$ is an open cover of K which is almost compact. Therefore, there exists a finite subfamily $\{U_{y_i},i=1,2,...n\}$ such that $K\subset\bigcup_{i=1}^{n}cl(U_{y_i})$ .Let $G=\bigcap_{i=1}^{n}U_{x_i}$ Now G is an open set containing x and $G\cap K=\phi$ implying thereby that $x\notin clK$. Hence K is closed.

Theorem 2.3. Let $f:X\rightarrow Y$ be continuous. Then image of every almost compact subset of X is almost-compact.

Proof. Let K be any almost-compact subset of X. To prove f(K) is almost-compact, let $\{V_\alpha\}$ be any open cover of f(K).Then $f^{-1}(V_\alpha)$ is an open cover of K which is almost compact. Therefore there exists a finite subfamily $f^{-1}(V_{\alpha_i})$ i=1,2,---n such that $K=\bigcup_{i=1}^{n}cl(f^{-1}V_{\alpha_i})$.

Now $f(K)=f(\bigcup_{i=1}^{n}cl(f^{-1}V_{\alpha_i}))=\bigcup_{i=1}^{n}f(cl(f^{-1}(V_{\alpha_i}))\subset\bigcup_{i=1}^{n}cl(V_{\alpha_i})$ as $V_{\alpha_i}\subset cl(V_{\alpha_i})$, $f^{-1}(V_{\alpha_i})\subset f^{-1}(cl(V_{\alpha_i}))\Rightarrow cl\ f^{-1}(V_{\alpha_i})\subset f^{-1}(cl(V_{\alpha_i}))$ therefore $f(\ clf^{-1}(V_{\alpha_i}))\subset f(f^{-1}(cl(V_{\alpha_i})))\subset cl(V_{\alpha_i})$.

Combining above lemma 2.1 and theorem 2.3,we get the following

Theorem 2.4. Let $f:X\rightarrow Y$ be continuous, where X is almost compact and Y is T_2. Then f is almost closed.

Proof. The proof follows from above lemma 2.1 ,theorem 2.3 and the fact that a regular closed subset of almost compact space is almost compact.

Combining theorems 1.3, 2.2 and 2.4 we get the following

Theorem 2.5. Let $f:X\rightarrow Y$ be any map with(closed) compact fibers, where X is (regular)T_2 and almost compact, Y is T_2 and one of the conditions (a),(b),(c) in theorem 1.3 is satisfied, then the following conditions (i) to (iii) are equivalent.

(i) f is continuous

(ii) f is almost closed

(iii) f has closed graph.

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