Fixed points of self maps in \( d_p \) – complete topological spaces

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Abstract: The purpose of this paper is to prove some fixed point theorems in \( d_p \)-complete topological spaces which generalize the results of Troy L. Hicks and B.E. Rhoades\([6]\).

Keywords: \( d_p \)-complete topological spaces, \( d \)-complete topological spaces, orbitally lower semi continuous and orbitally continuous maps.

I. Introduction

In 1992, Troy L. Hicks\([5]\) introduced the notion of \( d \)-complete topological spaces as follows:

1.1 Definition: A topological space \((X, t)\) is said to be \( d \)-complete if there is a mapping \( d : X \times X \rightarrow [0, \infty) \) such that (i) \( d(x, y) = 0 \Leftrightarrow x = y \) and (ii) \( < x_n > \) is a sequence in \( X \) such that \( \sum_{n=1}^{\infty} d(x_n, x_{n+1}) \) is convergent implies that \( < x_n > \) converges in \((X, t)\).

Troy L. Hicks and B.E. Rhoades\([6]\) proved the following theorem in \( d \)-complete topological spaces.

1.2 Theorem: Let \( T \) be a self map of a topological space \((X, t)\) and \( d : X \times X \rightarrow [0, \infty) \) such that \( O_T(u) \) has a cluster point \( z \in X \). If

- a) \( G(x) = d(x, Tx) \) is \( T \)-orbitally continuous at \( z \) and \( Tz \)
- b) \( T \) is \( d \)-orbitally continuous at \( z \) and
- c) \( d(Tx, T^2x) < d(x, Tx) \) for all \( x, T \in O_T(u) \) then \( Tz = z \).

In this paper we introduce \( d_p \)-complete topological spaces as a generalization of \( d \)-complete topological spaces. In fact, we define \( d_p \)-complete topological spaces for any integer \( p \geq 2 \). For a non-empty set \( X \), let \( X^p \) be its \( p \)-fold cartesian product.

1.3 Definition: A topological space \((X, t)\) is said to be \( d_p \)-complete if there is a mapping \( d_p : X^p \rightarrow [0, \infty) \) such that (i) \( d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p \) and (ii) \( < x_n > \) is a sequence in \( X \) with

\[
\lim_{n \rightarrow \infty} d_p(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+p-1}) = 0
\]

implies that \( < x_n > \) converges to some point in \((X, t)\). A \( d_p \)-complete topological space is denoted by \((X, t, d_p)\).

1.4 Remark: The function \( d \) in the Definition 1.1 and the function \( d_2 \) (the case \( p = 2 \)) in Definition 1.2 are both defined on \( X \times X \) and satisfy condition (i) of the definitions which are identical. Since the convergence of an infinite series \( \sum_{n=1}^{\infty} \alpha_n \) of real numbers implies that \( \lim_{n \rightarrow \infty} \alpha_n = 0 \), but not conversely; it follows that every \( d \)-complete topological space is \( d_2 \)-complete, but not conversely. Therefore the class of \( d_2 \)-complete topological spaces is wider than the class of \( d \)-complete spaces and hence a separate study of fixed point theorems of self-maps on \( d_2 \)-complete topological spaces is meaningful.

The purpose of this paper is to establish fixed point theorems of self-maps of \( d_p \)-complete topological spaces for \( p \geq 2 \).

II. Preliminaries

Let \( X \) be a non-empty set. A mapping \( d_p : X^p \rightarrow [0, \infty) \) will be called a \( p \)-non-negative on \( X \) provided \( d_p(x_1, x_2, \ldots, x_p) = 0 \Leftrightarrow x_1 = x_2 = \ldots = x_p \).
2.1 **Definition:** Suppose \((X, t)\) is a topological space and \(d_p\) is a \(p\)-non-negative on \(X\). A sequence \(<x_n>\) in \(X\) is said to be a \(d_p\)-Cauchy sequence if \(d_p(x_n, x_{n+1}, \ldots, x_{n+p-1}) \to 0\) as \(n \to \infty\).

In view of Definition 2.1, a topological space \((X, t)\) is \(d_p\)-complete if there is a \(p\)-non-negative \(d_p\) on \(X\) such that every \(d_p\)-Cauchy sequence in \(X\) converges to some point in \((X, t)\).

If \(T\) is a self map of a non-empty set \(X\) and \(x \in X\), then the orbit of \(x\), \(O_T(x)\) is given by \(O_T(x) = \{x, Tx, T^2x, \ldots\}\). If \(T\) is a self map of a topological space \(X\), then a mapping \(G : X \to [0, \infty]\) is said to be \(T\)-orbitally lower semi-continuous (resp. \(T\)-orbitally continuous) at \(x^* \in X\) if \(<x_n>\) is a sequence in \(O_T(x)\) for some \(x \in X\) with \(x_n \to x^*\) as \(n \to \infty\) then \(G(x^*) \leq \liminf G(x_n)\) (resp. \(G(x^*) = \lim G(x_n)\)). A self map \(T\) of topological space \(X\) is said to be \(w\)-continuous at \(x \in X\) if \(x_n \to x\) as \(n \to \infty\) implies \(Tx_n \to Tx\) as \(n \to \infty\).

If \(d_p\) is a \(p\)-non-negative on a non-empty set \(X\), and \(T : X \to X\) then we write, for simplicity of notation, that
\[
(2.2) \quad G_p(x) := d_p(x, Tx, T^2x, \ldots, T^{p-1}x) \quad \text{for} \quad x \in X
\]
Clearly we have
\[
(2.3) \quad G_p(x) = 0 \quad \text{if and only if} \quad x \quad \text{is a fixed point of} \quad T.
\]

### III. Main results

3.1 **Theorem:** Suppose \(T\) is a self-map of a topological space \((X, t)\) and \(d_p\) is a \(p\)-non-negative on \(X\).

Suppose that there is a \(u \in X\) such that \(O_T(u)\) has a cluster point \(z \in X\). If

a) \(G_p(x)\) is \(T\)-orbitally continuous at \(z\) and \(Tz\)

b) \(T\) is orbitally continuous at \(z\), and

c) \(G_p(Tz) < G_p(x)\) for all \(x, Tx \in O_T(u)\) (the closure of \(O_T(u)\)), then \(Tz = z\).

**Proof:** Let \(a_i = G_p(T^iu)\) for \(i \geq 1\). Then, by (c), we get \(a_{i+1} < a_i\) and therefore \(\lim_{n \to \infty} a_i = \alpha\) exists and in fact, \(\alpha = \inf_{i} a_i\). Since \(z\) is a cluster point of \(O_T(u)\), there is a sequence \(<T^ku>\) in \(O_T(u)\) such that \(T^ku \to z\) as \(k \to \infty\). Therefore, by (a), we have
\[
(3.2) \quad G_p(z) = \lim_{k \to \infty} a_k
\]

Also, it follows from (b) that \(T^{i+1}u = T(T^iu) \to Tz\) as \(k \to \infty\) and since \(G_p(x)\) is \(T\)-orbitally continuous at \(Tz\) we get
\[
(3.3) \quad G_p(Tz) = \lim_{k \to \infty} a_{i+1}
\]

Now (3.2) and (3.3) imply that \(G_p(Tz) = G_p(z)\) which forces \(Tz = z\) (For if \(Tz \neq z\), then (c) gives \(G_p(Tz) < G_p(z)\)).

### IV. Consequences

To present certain consequences of the main result, we introduce some notations:

If \(d_p\) is a \(p\)-non-negative on a non-empty set \(X\) and \(T\) is a self-map of \(X\), then for any \(x, y \in X\) we write
\[
(4.1) \quad H_p(x, y) = d_p(x, y, Ty, T^2y, \ldots, T^{p-1}y)
\]
\[
(4.2) \quad E_p(x, y) = d_p(x, y, y, \ldots, y)
\]
Clearly
\[
(4.3) \quad H_p(x, Tx) = G_p(x) \quad \text{and} \quad E_p(x, x) = 0.
\]

4.4 **Theorem:** Let \(T\) be a self-map of a topological space \((X, t)\) and \(d_p\) be a \(p\)-non-negative on \(X\). Suppose that there is a \(u \in X\) such that \(O_T(u)\) has a cluster point \(z \in X\). If

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a) \( G_p(x) \) is \( T \)-orbitally continuous at \( z \) and \( Tz \)

b) \( T \) is orbitally continuous at \( z \), and

c) \( H_p(Tx,Ty) \leq \frac{M_1(x)}{M_2(x)} \), where

\[ M_1(x) = \max \left\{ H_p(x,y), H_p(x,Ty), H_p(x,y), E_p(y,Tx), G_p(x), H_p(x,Ty), G_p(y), E_p(y,Tx) \right\} \]

and \( M_2(x) = \max \left\{ H_p(x,Ty), E_p(y,Tx) \right\} \) for all \( x, y \in X \) with \( x \neq Ty \) or \( y \neq Tx \), then \( Tz = z \).

**Proof:** Taking \( y = Tx \) in the inequality of the theorem and using (4.3), we get

\[ G_p(Tx) \leq \frac{\max \left\{ G_p(x), H_p(x,T^2x), G_p(x), H_p(x,T^2x) \right\}}{H_p(x,T^2x)} \]

\[ \Rightarrow G_p(Tx) \leq \frac{G_p(x), H_p(x,T^2x)}{H_p(x,T^2x)} \]

\[ \Rightarrow G_p(Tx) \leq G_p(x) \]

and therefore the theorem follows from Theorem 3.1.

**4.5 Remark:** Note that, the result of Hicks and Rhoades ([6], Corollary 3, pp.849) is a particular case of Theorem 4.4 and the corresponding result for metric spaces has been proved by Achari ([1], Theorem 1).

**4.6 Theorem:** Let \( T \) be a self-map of a topological space \((X, t)\) and \( d_p \) be a \( p \)-non-negative on \( X \). Suppose that there is a \( u \in X \) such that \( O_T(u) \) has a cluster point \( z \in X \). If

a) \( G_p(x) \) is \( T \)-orbitally continuous at \( z \) and \( Tz \)

b) \( T \) is orbitally continuous at \( z \), and

c) \( H_p(Tx,Ty) < \max \left\{ H_p(x,y), \frac{G_p(x), G_p(y)}{H_p(x,y)}, A(x,y), H_p(x,y), E_p(y,Tx) \right\} \)

for all \( x, y \in X \) with \( x \neq y \), where \( A(x,y) = a(x,y,Ty, \ldots, T^{p-2}y) \) and \( a : X^p \to [0, \infty) \). Then \( Tz = z \).

**Proof:** Taking \( y = Tx \) in the inequality of the theorem and using (4.3), we get

\[ G_p(Tx) < \max \left\{ \frac{G_p(x), G_p(Tx)}{G_p(x)} \right\} \]

\[ \Rightarrow G_p(Tx) < \frac{G_p(x), G_p(Tx)}{G_p(x)} \]

which implies \( G_p(Tx) < G_p(x) \) and therefore the theorem follows from Theorem 3.1.

**4.7 Remark:** Note that, the result of Hicks and Rhoades ([6], Corollary 4, pp.849) is a particular case of Theorem 4.6 and the corresponding result for metric spaces has been first proved by L. B. Ciric ([2], Theorem 2).

**4.8 Theorem:** Let \( T \) be a self-map of a topological space \((X, t)\) and \( d_p \) be a \( p \)-non-negative on \( X \). Suppose that there is a \( u \in X \) such that \( O_T(u) \) has a cluster point \( z \in X \). If

a) \( G_p(x) \) is \( T \)-orbitally continuous at \( z \) and \( Tz \)

b) \( T \) is orbitally continuous at \( z \), and

c) \( H_p(Tx,Ty)H_p(x,y) < H_p(x,y) \left\{ a_1H_p(x,y) + a_2G_p(x) + a_3G_p(y) + a_4E_p(y,Tx) \right\} \]

\[ + a_5G_p(x)G_p(y) \]

for all \( x, y \in X \), where \( a_i \geq 0 \) and \( \sum_{i=1}^{5} a_i = 1 \). Then \( Tz = z \) and \( z \) is unique.

**Proof:** Taking \( y = Tx \) in the inequality of the theorem and using (4.3), we get
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$$G_p(Tx) G_p(x) < G_p(x) [a_1 G_p(x) + a_2 G_p(Tx) + a_3 G_p(Tx)] + a_5 G_p(x) G_p(Tx)$$

$$\Rightarrow G_p(Tx) G_p(x) < a_1 [G_p(x)]^2 + a_2 [G_p(x)]^2 + a_5 G_p(x) G_p(Tx) + a_3 G_p(Tx) G_p(Tx)$$

$$\Rightarrow G_p(Tx) G_p(x) < (a_1 + a_2) [G_p(x)]^2 + (a_3 + a_5) G_p(x) G_p(Tx)$$

$$\Rightarrow (1 - a_5) G_p(x) G_p(Tx) < (a_1 + a_2) G_p(x)$$

$$\Rightarrow G_p(Tx) < \frac{a_1 + a_2}{1 - a_3 - a_5} G_p(x)$$

which gives $G_p(Tx) < G_p(x)$ and therefore the theorem follows from Theorem 3.1.

4.9 Remark: Note that, the result of Hicks and Rhoades ([6], Corollary 5, pp.849) is a particular case of Theorem 4.8 and the corresponding result for metric spaces has been first proved by K.M. Ghosh ([4], Theorem 2).

References