

Solitons Solutions to Nonlinear Partial Differential Equations by the Tanh Method

YusurSuhail Ali

Computer Science Department, Al-Rafidain University College, Iraq

Abstract: The Tanh method is implemented for the exact solutions of some different kinds of nonlinear partial differential equations. New solutions for nonlinear equations such as Benjamin-Bona-Mahony (BBM) equation, Gardner equation, Cassama-Holm equation, and two component Kdv evolutionary system are obtained.

Keywords: Tanh method, Benjamin-Bona-Mahony (BBM) equation, Gardner equation, Cassama-Holm equation

I. INTRODUCTION

Non-linear evolution equations (NLEEs) had come a long way through. These NLEEs appear in various areas of Physics, Engineering, Biological Sciences, Geological Sciences and many other places [1–10]. These equations arise of necessity. Subsequently they are studied in these various scientific contexts. There are various aspects of these NLEEs that are studied by various scientists and engineers as the need arises. Some of the commonly studied aspects are integrability, conservation laws, numerical solutions and many other aspects.

II. OUTLINE OF THE TANH METHOD:

The method is applied to find out exact solutions of a coupled system of nonlinear differential equation with unknown:

$$P(u, u_x, u_t, u_{xx}, \dots) = 0 \quad (1)$$

where P is a polynomial of the variable u and its derivatives. If we consider $u(x, t) = u(\xi)$, and $\xi = kx + \lambda t$, so that $u(x, t) = U(\xi)$, we can use the following changes:

$$\frac{\partial}{\partial t} = \lambda \frac{d}{d\xi}, \quad \frac{\partial}{\partial x} = k \frac{d}{d\xi}, \quad \frac{\partial^2}{\partial x^2} = k^2 \frac{d^2}{d\xi^2}, \quad \frac{\partial^3}{\partial x^3} = k^3 \frac{d^3}{d\xi^3} \quad (2)$$

and so on, then Eq. (1) becomes an ordinary differential equation

$$Q(U, U', U'', U''', \dots) = 0 \quad (3)$$

with Q being another polynomial form, which will be called the reduced ordinary differential equation.

Integrating Eq.(3) as long as all terms contain derivatives, the integration constants are considered to be zeros in view of the localized solutions. However, the nonzero constants can be used and handled as well. Now finding the traveling wave solution to Eq.(1) is equivalent to obtaining the solution to the reduced ordinary differential equation (3).

The tanh method is introduced by Malfliet and Wazwaz [1,3, and 4]. It is based on a priori assumption that the traveling wave solutions can be expressed in terms of the tanh function to solve the coupled KdV equations. For the tanh method, we introduce the new independent variable

$$Y(x, t) = \tanh(\xi) \quad (4)$$

that leads to the change of variables:

$$\begin{aligned} \frac{d}{d\xi} &= (1 - Y^2) \frac{d}{dY} \\ \frac{d^2}{d\xi^2} &= -2Y(1 - Y^2) \frac{d}{dY} + (1 - Y^2)^2 \frac{d^2}{dY^2} \\ \frac{d^3}{d\xi^3} &= 2(1 - Y^2)(3Y^2 - 1) \frac{d}{dY} - 6Y(1 - Y^2)^2 \frac{d^2}{dY^2} + (1 - Y^2)^3 \frac{d^3}{dY^3} \end{aligned} \quad (5)$$

Then the solution is expressed in the form

$$u(x, t) = U(\xi) = \sum_{i=0}^m a_i Y^i \quad (6)$$

where the parameter m can be found by balancing the highest-order linear term with the nonlinear terms in Eq.(3), and $k, \lambda, a_0, a_1, \dots, a_m$ are to be determined.

Substituting Eq.(4) or Eq.(5) into Eq.(3) will yield a set of algebraic equations for $k, \lambda, a_0, a_1, \dots, a_m$ because all coefficients of Y^i have to vanish for $i=0, 1, 2, 3, \dots$. From these relations $k, \lambda, a_0, a_1, \dots, a_m$ can be obtained. Having determined these parameters, we can obtain the analytic solution $u(x, t)$ in a closed form. These methods seem to be powerful tool in dealing with coupled nonlinear physical models.

III. APPLICATIONS:

1. Benjamin-Bona-Mahony (BBM) equation:

Consider the following BBM equation [11],

$$u_t = u_{xxt} - u_x - uu_x \quad (7)$$

Let $u(x, t) = u(\xi)$ and by using the wave variable $\xi = kx + \lambda t$, equation (7) turns to be the following ordinary differential equation:

$$\lambda u' - \lambda k^2 u''' + ku' + ku u' = 0 \quad (8)$$

by integrating equation (8) once with zero constant, we have

$$(\lambda + k)u - \lambda k^2 u'' + \frac{k}{2}u^2 = 0 \quad (9)$$

Applying Tanh Method, eq.(9) becomes

$$(\lambda + k)u - \lambda k^2 \left[-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2u}{dY^2} \right] + \frac{k}{2}u^2 = 0 \quad (10)$$

Where the solution can be formed as follows:

$$u(x, t) = \sum_{i=0}^m a_i Y^i \quad (11)$$

Where the parameter m can be determined by balancing the highest order of linear and non-linear terms in equation (3), which gives m=2, then equation (11) will become:

$$u(x, t) = \sum_{i=0}^2 a_i Y^i$$

$$u = a_0 + a_1 Y + a_2 Y^2, \quad a_2 \neq 0 \quad (12)$$

Where a_0, a_1 and a_2 are constant parameters that must be determined; and to calculate those parameters substitute equation u and u' from equation (11) in equation (10), and equating the terms with identical power of the parameter Y , then we obtain the following system of equations:

$$Y^0: (\lambda + k)a_0 - 2\lambda k^2 a_2 + \frac{k}{2}a_0^2 = 0$$

$$Y^1: (\lambda + k)a_1 + ka_0 a_1 = 0$$

$$Y^2: (\lambda + k)a_2 + \frac{k}{2}(2a_0 a_2 + a_1^2) = 0$$

$$Y^3: ka_1 a_2 = 0$$

$$Y^4: \frac{k}{2}a_2^2 = 0 \quad (13)$$

Solving system of equations (13), we get:

$$a_0 = -\frac{(\lambda+k)}{k}, \quad a_1 = 0, \quad a_2 = -\frac{(\lambda+k)^2}{4\lambda k^3}$$

And by substituting those parameters in equation (3.2), we get the solution:

$$u(x, t) = -\frac{(\lambda+k)}{k} \left[1 + \frac{(\lambda+k)}{4\lambda k^2} \tanh^2(kx + \lambda t) \right] \quad (14)$$

Now, for $\lambda = k = 1$, we have:

$$u(x, t) = -2 \left[1 + \frac{1}{2} \tanh^2(x + t) \right] \quad (15)$$

Figure (1) shows the behavior of the solution $u(x, t)$ in (15) for BBM equation.

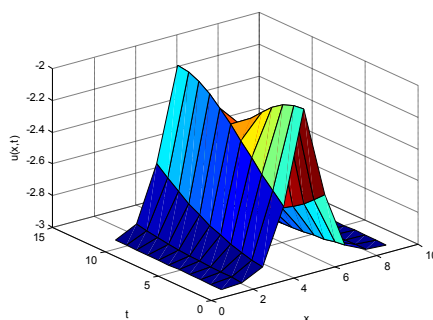


Fig.(1) solution $u(x, t)$ in (15) for BBM equation.

2. Gardner equation:

Consider the following Gardner equation[11],

$$u_t = u_{xxx} + 6uu_x \quad (16)$$

Let $u(x, t) = u(\xi)$ and by using the wave variable $\xi = kx + \lambda t$, equation (16) turns to be the following ordinary differential equation:

$$\lambda u' - k^3 u''' - 3k(u^2)' = 0 \quad (17)$$

And by integrating equation (4.1) once with zero constant, we have

$$\lambda u - k^3 u'' - 3ku^2 = 0 \quad (18)$$

Applying Tanh Method, equation (18) becomes:

$$\lambda u - k^3 \left[-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2u}{dY^2} \right] + 3ku^2 = 0 \quad (19)$$

Where the solution can be formed as follows:

$$u(x, t) = \sum_{i=0}^m a_i Y^i \quad (20)$$

Where the parameter m can be determined by balancing the highest order of linear and non-linear terms in equation (19), which gives m=2, then equation (20) will becomes:

$$u = a_0 + a_1 Y + a_2 Y^2, \quad a_2 \neq 0 \quad (21)$$

Where a₀, a₁ and a₂ are constant parameters that must be determined; and to calculate those parameters substitute equation u and u' from equation (21) in equation (19), and equating the terms with identical power of the parameter Y, then we obtain:

$$\begin{aligned} Y^0: \lambda a_0 - 2k^3 a_2 - 3ka_0^2 &= 0 \\ Y^1: \lambda a_1 - 6ka_0 a_1 &= 0 \\ Y^2: \lambda a_2 - 6ka_0 a_2 - 3ka_1^2 &= 0 \\ Y^3: -6ka_1 a_2 &= 0 \end{aligned}$$

$$Y^4: -3ka_2^2 = 0 \quad (22)$$

Solving system of equations (22), we get:

$$a_0 = \frac{\lambda}{6k}, \quad a_1 = 0, \quad a_2 = \frac{\lambda^2}{24k^4}$$

by substituting those parameters in equation (23), we get the solution:

$$u(x, t) = \frac{\lambda}{6k} \left[1 + \frac{\lambda}{4k^3} \tanh^2(kx + \lambda t) \right] \quad (23)$$

Now, for $\lambda = k = 1$, we have:

$$u(x, t) = \frac{1}{6} \left[1 + \frac{1}{4} \tanh^2(x + t) \right] \quad (24)$$

Figure (2) shows the behavior of the solution u(x,t) in (24) for Gardner equation.

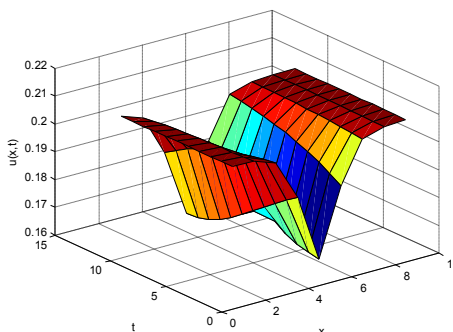


Fig.(2) the solution u(x,t) in (24) for Gardner equation.

3. Cassama-Holm equation:

Consider the following Cassama-Holm equation [11],

$$u_t + 2\mu u_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0 \quad (25)$$

Let $u(x, t) = u(\xi)$ and by using the wave variable $\xi = kx + \lambda t$, equation (25) turns to be the following ordinary differential equation:

$$(\lambda + 2\mu k)u' + \frac{3}{2}k(u^2)' - \lambda k^2 u''' - k^3 \left[\frac{1}{2}(u')^2 + uu'' \right]' = 0 \quad (26)$$

by integrating equation (26) once with zero constant, we have

$$(\lambda + 2\mu k)u + \frac{3}{2}ku^2 - \lambda k^2 u'' - k^3 \frac{1}{2}(u')^2 - k^3 uu'' = 0 \quad (27)$$

Applying Tanh Method, equation (27) becomes:

$$(\lambda + 2\mu k)u + \frac{3}{2}ku^2 - \lambda k^2 \left[-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2u}{dY^2} \right] - \frac{k^3}{2} \left[(1 - Y^2) \frac{du}{dY} \right]^2 - k^3 u \left[-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2u}{dY^2} \right] = 0 \quad (28)$$

Where the solution can be formed as follows:

$$u(x, t) = \sum_{i=0}^m a_i Y^i \quad (29)$$

Where the parameter m can be determined by balancing the highest order of linear and non-linear terms in equation (28), which gives m=2, then equation (29) will becomes:

$$u = a_0 + a_1 Y + a_2 Y^2, \quad a_2 \neq 0 \quad (30)$$

Where a_0, a_1 and a_2 are constant parameters that must be determined; and to calculate those parameters substitute equation (30), u' and u'' from equation (30) in equation (28), and equating the terms with identical power of the parameter Y , then we obtain:

$$\begin{aligned} Y^0: (\lambda + 2\mu k)a_0 + \frac{3}{2}ka_0^2 - 2\lambda k^2 a_2 - \frac{k^3}{2}a_1^2 - 2k^3 a_0 a_2 &= 0 \\ Y^1: (\lambda + 2\mu k)a_1 + 3ka_0 a_1 - 4k^3 a_1 a_2 &= 0 \\ Y^2: (\lambda + 2\mu k)a_2 + 3ka_0 a_2 + \frac{3}{2}ka_1^2 - 4k^3 a_2^2 &= 0 \\ Y^3: 3ka_1 a_2 &= 0 \end{aligned}$$

$$Y^4: \frac{3}{2}ka_2^2 = 0 \quad (31)$$

Solving system of equations (31), we get:

$$a_0 = \frac{\lambda^2 + 2\lambda\mu k}{2k(\mu k - \lambda)}, \quad a_1 = 0, \quad a_2 = \frac{(\lambda + 2\mu k)^2}{8k^3(\mu k - \lambda)}, \quad \mu k > \lambda$$

by substituting those parameters in equation (30), we get the solution:

$$u(x, t) = \frac{(\lambda + 2\mu k)}{2k(\mu k - \lambda)} \left[\lambda + \frac{(\lambda + 2\mu k)}{4k^2} \tanh^2(kx + \lambda t) \right] \quad (32)$$

Now, for $\lambda = k = 1$ and $\mu = \frac{3}{2}$, we have:

$$u(x, t) = 4[1 + \tanh^2(x + t)] \quad (33)$$

Figure (3) shows the behavior of the solution $u(x, t)$ in (33) for Cassama-Holm equation.

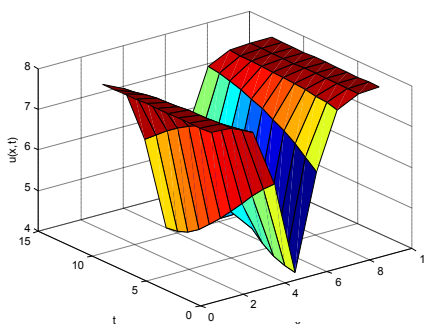


Fig.(3) the solution $u(x, t)$ in (33) for Cassama-Holm equation.

4. Two component Kdv evolutionary system [11]:

Consider the following two component evolutionary system of homogeneous Kdv equations of order 2,

$$u_t = -3v_{xx} \quad (34)$$

$$v_t = u_{xx} + 4u^2 \quad (35)$$

Let $u(x, t) = u(\xi)$ and $v(x, t) = v(\xi)$ and by using the wave variable $\xi = kx + \lambda t$, equations in (34), and (35) turn to be the following ordinary differential system:

$$\lambda u' = -3k^2 v'' \quad (36)$$

$$\lambda v' = k^2 u'' + 4u^2 \quad (37)$$

by integrating equation (36) once with zero constant, we have

$$\lambda u = -3k^2 v \quad (38)$$

that leads to:

$$v' = \frac{\lambda}{-3k^2} u \quad (39)$$

Now substitute in equation (39) in (37), we get:

$$-\frac{\lambda^2}{3k^2} u - k^2 u'' - 4u^2 = 0 \quad (40)$$

Applying Tanh Method, equation (40), becomes:

$$-\frac{\lambda^2}{3k^2} u - k^2 \left[-2Y(1 - Y^2) \frac{du}{dY} + (1 - Y^2)^2 \frac{d^2 u}{dY^2} \right] - 4u^2 = 0 \quad (41)$$

Where the solution can be formed as follows:

$$u(x, t) = \sum_{i=0}^m a_i Y^i \quad (42)$$

Where the parameter m can be determined by balancing the highest order of linear and non-linear terms in equation (41), which gives $m=2$, then :

$$u = a_0 + a_1 Y + a_2 Y^2, \quad a_2 \neq 0 \quad (43)$$

Where a_0 , a_1 and a_2 are constant parameters that must be determined; and to calculate those parameters substitute equation (43) and u' from equation (43) in equation (41), and equating the terms with identical power of the parameter Y , then we obtain:

$$Y^0: -\frac{\lambda^2}{3k^2}a_0 - 2k^2a_2 - 4a_0^2 = 0$$

$$Y^1: -\frac{\lambda^2}{3k^2}a_1 - 8a_0a_1 = 0$$

$$Y^2: -\frac{\lambda^2}{3k^2}a_2 - 8a_0a_2 - 4a_1^2 = 0$$

$$Y^3: -8a_1a_2 = 0$$

$$Y^4: -4a_2^2 = 0 \quad (44)$$

Solving system of equations (44), we get:

$$a_0 = -\frac{\lambda^2}{24k^2}, \quad a_1 = 0, \quad a_2 = \frac{\lambda^4}{288k^6}$$

substituting those parameters in equation (43), we get the solution:

$$u(\beta) = \frac{\lambda^2}{24k^2} \left[-1 + \frac{\lambda^2}{12k^4} \tanh^2(\beta) \right] \quad (45)$$

substituting equation (45) in equation (39) and integrating once with respect to ξ , we have:

$$v(\xi) = -\frac{\lambda^3}{72k^4} \left[-\xi + \frac{\lambda^2}{12k^4} (\xi - \tanh \xi) \right] + c \quad (46)$$

Then the solutions of $u(x,t)$ and $v(x,t)$ will become:

$$u(x,t) = \frac{\lambda^2}{24k^2} \left[-1 + \frac{\lambda^2}{12k^4} \tanh^2(kx + \lambda t) \right] \quad (47)$$

And

$$v(x,t) = -\frac{\lambda^3}{72k^4} \left[-(kx + \lambda t) + \frac{\lambda^2}{12k^4} [kx + \lambda t - \tanh(kx + \lambda t)] \right] + c \quad (48)$$

Now, for $\lambda = k = 1$, we have:

$$u(x,t) = \frac{1}{24} \left[-1 + \frac{1}{12} \tanh^2(x+t) \right] \quad (49)$$

$$v(x,t) = -\frac{1}{72} \left[-(x+t) + \frac{1}{12} [(x+t) - \tanh(x+t)] \right] + c \quad (50)$$

Figures (4) and (5) show the behavior of the solution of $u(x,t)$ and $v(x,t)$ in (49) and (50) respectively for the Kdv system.

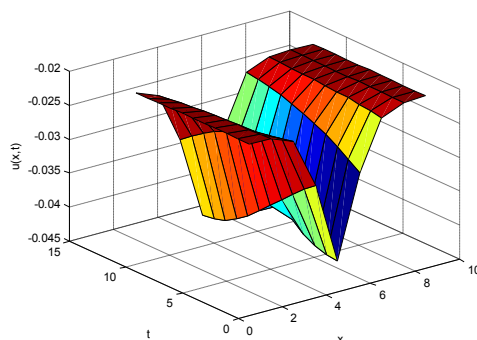


Fig.(4) the solution of $u(x,t)$ in (49) for the Kdv system.

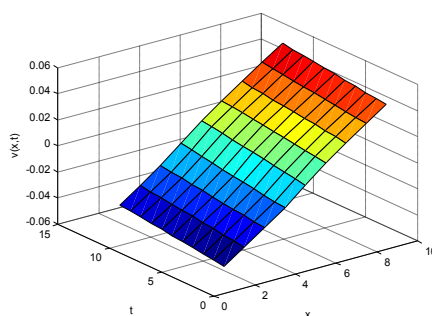


Fig.(5) the solution of $v(x,t)$ in (50) for the Kdv system.

IV. CONCLUSION

In this paper, the tanh method has been successfully applied to find the solution for four nonlinear partial differential equations such as the Benjamin-Bona-Mahony (BBM) equation, Gardner equation, Cassama-Holm equation, and two component Kdv evolutionary system of equations. The tanh method is used to find a new complex travelling wave solutions. The results show that the tanh method is a powerful mathematical tool to solve the nonlinear PDEs.

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