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A Continuous Block Integrator for the Solution of Stiff and Oscillatory Differential Equations

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Abstract: *In this paper, we present a continuous block integrator for direct integration of stiff and oscillatory first-order ordinary differential equations using interpolation and collocation techniques. The approximate solution used in the derivation is a combination of power series and exponential function. The paper further investigates the properties of the block integrator and found it to be zero-stable, consistent and convergent. The integrator was also tested on some sampled stiff and oscillatory problems and found to perform better than some existing ones.*
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I. Introduction

Nowadays, the integration of Ordinary Differential Equations (ODEs) could be carried out using block integrators. In this paper, we present a continuous block integrator for direct integration of stiff and oscillatory problems of the form,

$$y' = f(x, y),\ y(a) = y_0,\ x \in [a, b] \qquad (1)$$

where $f : \Re \times \Re^m \to \Re^m$, $y, y_0 \in \Re^m$, f satisfies Lipchitz condition which guarantees the existence and uniqueness of solution of (1). The development of numerical integration formulas for stiff as well as oscillatory differential equations has attracted considerable attention in the past (Fatunla, 1980). A special problem arising in the solution of ODEs is stiffness. This problem occurs in single linear and nonlinear ODEs, higher-order linear and nonlinear ODEs and systems of linear and nonlinear ODEs (Hoffman, 2001). It is also important to note that mathematical models of physical situations in kinetic chemical reactions, process control and electrical circuit theory often results to stiff ODEs (Fatunla, 1980). According to Sanugi and Evans (1989), an interesting and important class of IVPs which can also arise in practice consists of differential equations whose solutions are known to be periodic or to oscillate with a known frequency. Examples of such problems can be found in the field of ecology, medical sciences and oscillatory motion in a nonlinear force field.

Almost invariably, most conventional numerical integration solvers cannot efficiently cope with stiff and oscillatory problems of the form (1) as they lack adequate stability characteristics (Fatunla, 1980). The degree of stiffness of a problem depends on the definition of stiffness that is applied (Okunuga *et al.,* 2013). There are various definitions of stiffness in the literature as regards to ODEs. Lambert (1973) gave a simple definition of stiffness of an ODE in a such a manner that problem (1) possesses some stiffness if $\text{Re}(\lambda_i) < 0, i = 1(1)m$, where λ is the eigen value of the problem. A stiff equation is a differential equation for which certain numerical methods for solving the equation are numerically unstable, unless the step size is taken to be extremely small. The main idea is that the equation includes some terms that can lead to rapid variation in the solution. On the other hand, a nontrivial solution (function) of an ODE is called oscillating if it does not tend either to a finite limit or to infinity (i.e. if it has an infinite number of roots). The differential equation is called oscillating, if it has at least one oscillating solution (Borowski and Borwein, 2005). There are different concepts of the oscillation of a solution. The most widespread are oscillation at a point (usually taken to $+\infty$) and oscillation on an interval.

More recently, authors like Butcher (2003), Zarina, Mohammed, Kharil and Zanariah (2005), Awoyemi, Ademiluyi and Amuseghan (2007), Okunuga, Akinfewa and Daramola (2008), Abbas (2009), Areo, Ademiluyi, and Babatola (2011), Ibijola, Skwame and Kumleng (2011), Akinfewa, Yao and Jator (2011), Chollom, Olatunbosun and Omagu (2012), Okunuga, Sofoluwe and Ehigie (2013), Yakubu, Madaki and Kwami (2013), Ajie, Ikhile and Onumanyi (2013), among others, have all proposed block methods to generate numerical solution to (1). These authors proposed methods in which the approximate solution ranges from power series, Chebychev's, Lagrange's and Laguerre's polynomials.

In this paper, the derivation of the continuous block integrator is carried out using an approximate solution which is a combination of power series and exponential function.

II. Methodology: Construction of the Continuous Block Integrator

In deriving the integrator, interpolation and collocation procedures are used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s + r$ system of equations whose coefficients are determined by using appropriate procedures. The approximate solution to (1) is taken to be a combination of power series and exponential function given by,

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j + a_{r+s} \sum_{j=0}^{r+s} \frac{\alpha^j x^j}{j!} \tag{2}$$

with the first derivative given by,

$$y'(x) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=1}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!} \tag{3}$$

where $a_j, \alpha^j \in \mathbb{R}$ for $j = 0(1)4$ and $y(x)$ is continuously differentiable. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$, of the integration interval $[a, b]$ with a constant step-size h, given by, $h = x_{n+1} - x_n$, $n = 0, 1, \ldots, N$.

Then, substituting (3) in (1) gives,

$$f(x, y) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=1}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!} \tag{4}$$

Now, interpolating (2) at point $x_{n+s}, s = 0$ and collocating (4) at points $x_{n+r}, r = 0(1)3$, leads to the following system of equations,

$$AX = U \tag{5}$$

where

$$A = [a_0 \ a_1 \ a_2 \ a_3 \ a_4]^T$$

$$U = [y_n \ f_n \ f_{n+1} \ f_{n+2} \ f_{n+3}]^T$$

and

$$X = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & \left(1 + \alpha x_n + \frac{\alpha^2 x_n^2}{2!} + \frac{\alpha^3 x_n^3}{3!} + \frac{\alpha^4 x_n^4}{4!}\right) \\ 0 & 1 & 2x_n & 3x_n^2 & \left(\alpha + \alpha^2 x_n + \frac{\alpha^3 x_n^2}{2!} + \frac{\alpha^4 x_n^3}{3!}\right) \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & \left(\alpha + \alpha^2 x_{n+1} + \frac{\alpha^3 x_{n+1}^2}{2!} + \frac{\alpha^4 x_{n+1}^3}{3!}\right) \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & \left(\alpha + \alpha^2 x_{n+2} + \frac{\alpha^3 x_{n+2}^2}{2!} + \frac{\alpha^4 x_{n+2}^3}{3!}\right) \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & \left(\alpha + \alpha^2 x_{n+3} + \frac{\alpha^3 x_{n+3}^2}{2!} + \frac{\alpha^4 x_{n+3}^3}{3!}\right) \end{bmatrix}$$

Solving (5), for a_j's, $j = 0(1)4$ and substituting back into (2) gives a continuous linear multistep method of the form,

$$y(x) = \alpha_0(x) y_n + h \sum_{j=0}^{3} \beta_j(x) f_{n+j} \tag{6}$$

where $\alpha_0 = 1$ and the coefficients of f_{n+j} gives

$$\left.\begin{aligned}\beta_0 &= -\frac{1}{24}(t^4 - 8t^3 + 22t^2 - 24t)\\ \beta_1 &= \frac{1}{24}(3t^4 - 20t^3 + 26t^2)\\ \beta_2 &= -\frac{1}{24}(3t^4 - 16t^3 + 18t^2)\\ \beta_3 &= \frac{1}{24}(t^4 - 4t^3 + 4t^2)\end{aligned}\right\} \qquad (7)$$

where $t = (x - x_n)/h$. Evaluating (6) at $t = 1(1)3$ gives a continuous discrete block scheme of the form,

$$A^{(0)}\mathbf{Y}_m = \mathbf{E}\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \qquad (8)$$

where

$$\mathbf{Y}_m = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} \end{bmatrix}^T, \mathbf{y}_n = \begin{bmatrix} y_{n-2} & y_{n-1} & y_n \end{bmatrix}^T$$

$$\mathbf{F}(\mathbf{Y}_m) = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} \end{bmatrix}^T, \mathbf{f}(\mathbf{y}_n) = \begin{bmatrix} f_{n-2} & f_{n-1} & f_n \end{bmatrix}^T,$$

$$A^{(0)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad d = \begin{bmatrix} 0 & 0 & \frac{9}{24} \\ 0 & 0 & \frac{1}{3} \\ 0 & 0 & \frac{9}{24} \end{bmatrix} \quad b = \begin{bmatrix} \frac{19}{24} & \frac{-5}{24} & \frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{27}{24} & \frac{27}{24} & \frac{9}{24} \end{bmatrix}$$

III. Analysis of Basic Properties of the New Block Integrator

3.1. Order of the New Block Integrator

Let the linear operator $L\{y(x);h\}$ associated with the block (8) be defined as,

$$L\{y(x);h\} = A^{(0)}Y_m - Ey_n - hdf(y_n) - hbF(Y_m) \qquad (9)$$

expanding using Taylor series and comparing the coefficients of h gives,

$$L\{y(x);h\} = c_0 y(x) + c_1 hy'(x) + c_2 h^2 y''(x) + ... + c_p h^p y^p(x) + c_{p+1} h^{p+1} y^{p+1}(x) + ... \qquad (10)$$

Definition 3.1

The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if $c_0 = c_1 = c_2 = ... = c_p = 0$ *and* $c_{p+1} \neq 0$. c_{p+1} is called the error constant and the local truncation error is given by,

$$t_{n+k} = c_{p+1} h^{(p+1)} y^{(p+1)}(x_n) + O(h^{p+2}) \qquad (11)$$

For our method,

$$L\{y(x);h\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} - h \begin{bmatrix} \frac{9}{24} & \frac{19}{24} & -\frac{5}{24} & \frac{1}{24} \\ \frac{1}{3} & \frac{4}{3} & \frac{1}{3} & 0 \\ \frac{9}{24} & \frac{27}{24} & \frac{27}{24} & \frac{9}{24} \end{bmatrix} \begin{bmatrix} f_n \\ f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (12)$$

Expanding (12) in Taylor series gives,

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j - y_n - \frac{9h}{24}y_n^{'} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{19}{24}(1)^j - \frac{5}{24}(2)^j + \frac{1}{24}(3)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j - y_n - \frac{h}{3}y_n^{'} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{4}{3}(1)^j + \frac{1}{3}(2)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j - y_n - \frac{3h}{8}y_n^{'} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{9}{8}(1)^j + \frac{9}{8}(2)^j + \frac{3}{8}(3)^j\right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (13)$$

Hence, $\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = 0, \bar{c}_5 = \left[-2.64(-02), -1.11(-02), 3.75(-02)\right]^T$. Therefore, the block integrator is of order four.

3.2. Zero Stability

Definition 3.2

The block integrator (8) is said to be zero-stable, if the roots $z_s, s = 1, 2, ..., k$ of the first characteristic polynomial $\rho(z)$ defined by $\rho(z) = \det(z\mathbf{A}^{(0)} - \mathbf{E})$ satisfies $|z_s| \le 1$ and every root satisfying $|z_s| \le 1$ have multiplicity not exceeding the order of the differential equation. Moreover, as $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$ where μ is the order of the differential equation, r is the order of the matrices $\mathbf{A}^{(0)}$ *and* $\mathbf{E}$ (see Awoyemi *et al.* (2007) for details).

For our integrator,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0 \quad (14)$$

$\rho(z) = z^2(z-1) = 0, \Rightarrow z_1 = z_2 = 0, z_3 = 1$. Hence, the block integrator is zero-stable.

3.3. Convergence

The new block integrator is convergent by consequence of Dahlquist theorem below.

Theorem 3.1 (Dahlquist, 1956)

The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.

3.4. Region of Absolute Stability

Definition 3.3 (Yan, 2011)

Region of absolute stability is a region in the complex z plane, where $z = \lambda h$. It is defined as those values of z such that the numerical solutions of $y' = -\lambda y$ satisfy $y_j \to 0$ *as* $j \to \infty$ for any initial condition.

To determine the absolute stability regions of the block integrators, we adopt the boundary locus method. This is achieved by substituting the test equation,

$$y' = -\lambda y \quad (15)$$

into the block formula (8). This gives,

$$\mathbf{A}^{(0)}\mathbf{Y}_m(r) = \mathbf{E}\mathrm{y}_n(r) - h\lambda\mathbf{D}y_n(r) - h\lambda\mathbf{B}\mathbf{Y}_m(r) \quad (16)$$

Thus,

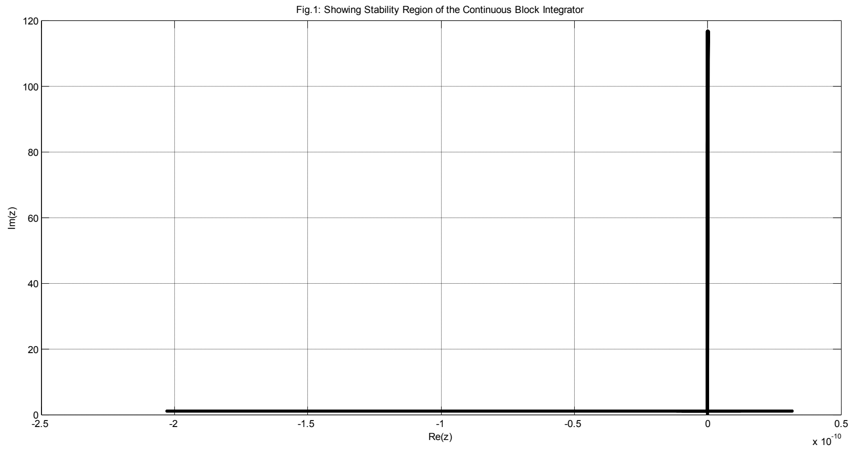
$$\bar{h}(r) = -\left(\frac{\mathbf{A}^{(0)}Y_m(r) - \mathbf{E}y_n(r)}{\mathbf{D}y_n(r) + \mathbf{B}Y_m(r)}\right) \quad (17)$$

Writing (17) in trigonometric ratios gives,

$$\bar{h}(\theta) = -\left(\frac{\mathbf{A}^{(0)}Y_m(\theta) - \mathbf{E}y_n(\theta)}{\mathbf{D}y_n(\theta) + \mathbf{B}Y_m(\theta)}\right) \quad (18)$$

where $r = e^{i\theta}$. Equation (18) is our characteristic or stability polynomial.

which gives the stability region shown in fig. 1 below.



According to Fatunla (1988), stiff algorithms have unbounded RAS. Also, Lambert (1973) showed that the stability region for L-stable schemes must encroach into the positive half of the complex z plane.

IV. Numerical Experiments

We shall use the following notations in the tables below;

ERR- |Exact Solution-Computed Result|

ESSI- Error in Skwame *et al.* (2012)

Problem 1

Consider the highly stiff ODE

$$y' = f(x, y) = -\alpha(y - F(x)) + F'(x), \quad y(x_0) = y_0 \tag{19}$$

which has the exact solution

$$y(x) = (y_0 - F(0))e^{-\alpha x} + F(x) \tag{20}$$

where α is a positive constant and $F(x)$ is a smooth slowly varying function. Equation (20) exhibits two widely different time scales: a rapidly changing term associated with $\exp(-\alpha x)$ and a slowly varying term associated with $F(x)$, (Gear, 1971).

Skwame *et al.* (2012) considered a special case of (19) where $\alpha = 10,\ F(x) = 0,\ x_0 = 0\, and\ y_0 = 1$. They solved the problem (19) by adopting an L-stable hybrid block Simpson's method of order six.

Problem 2

Consider the oscillatory ODE

$$y' = -\sin x - 200(y - \cos x),\ y(0) = 0 \tag{21}$$

whose exact solution is given by:

$$y(x) = \cos x - e^{-200x} \tag{22}$$

Though Yan (2011) did not solve this problem, he however observed that it has a solution that oscillates and grows exponentially in x. He further stated that most numerical methods do not perform well on this problem.

Table 1: Showing the result for stiff problem 1

x	Exact solution	Computed solution	ERR	ESSI
0.0100	0.9048374180359595	0.9048371857175926	2.323184e-007	6.28e-03
0.0200	0.8187307530779818	0.8187306524074074	1.006706e-007	1.88e-03
0.0300	0.7408182206817179	0.7408178956250000	3.250567e-007	3.26e-03
0.0400	0.6703200460356393	0.6703195798065542	4.662291e-007	1.06e-03
0.0500	0.6065306597126334	0.6065303190001389	3.407125e-007	3.85e-03
0.0600	0.5488116360940265	0.5488111544782535	4.816158e-007	1.45e-03
0.0700	0.4965853037914095	0.4965847405085258	5.632829e-007	5.02e-04
0.0800	0.4493289641172216	0.4493285145544429	4.495628e-007	2.76e-04
0.0900	0.4065696597405992	0.4065691245561065	5.351845e-007	1.01e-04
0.1000	0.3678794411714423	0.3678788624630128	5.787084e-007	3.74e-05

Table 2: Showing the result for oscillatory problem 2

x	Exact solution	Computed solution	ERR
0.0010	0.1812687469220599	0.1812753281481898	6.581226e-006
0.0020	0.3296779539650273	0.3296808918525185	2.937887e-006
0.0030	0.4511838639093485	0.4511932600033750	9.396094e-006
0.0040	0.5506630358934451	0.5506743405562253	1.130466e-005
0.0050	0.6321080588545993	0.6321159695640156	7.910709e-006
0.0060	0.6987877881417979	0.6987981014249823	1.031328e-005
0.0070	0.7533785361584351	0.7533889621198869	1.042596e-005
0.0080	0.7980714821760110	0.7980792802208783	7.798045e-006
0.0090	0.8346606120517877	0.8346691020538365	8.490002e-006
0.0100	0.8646147171800527	0.8646227560193114	8.038839e-006

V. Conclusion

We have presented a continuous block numerical integrator for the solution of stiff and oscillatory first-order ordinary differential equations. The approximate solution (basis function) adopted in this research produced a block integrator with L-stable stability region. This made it possible for the block integrator to perform well on stiff and oscillatory problems. The block integrator proposed was also found to be zero-stable, consistent and convergent. The integrator was also found to perform better than some existing methods.

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