Common Fixed Point Theorems for Sequence of Mappings in Generalisation of Strict Contractive Conditions

T.R. Vijayan,
Department of Mathematics, Pauyai College of Technology, Namakkal - 637018, India.

Abstract: The main purpose of this paper is to obtain fixed point theorems for sequence of mappings in strict contractive conditions which generalizes Theorem 1 of Aamri [1].

Key words and phrases: Fixed point, Coincidence point, compatible maps, weakly compatible map, non-compatible maps, property(E.A).

I. Introduction

In metric fixed point theory, strict contractive condition do not ensure the existence of common fixed point unless the space is assumed to be compact or the strict condition is replaced by stronger conditions as in [4-6]. In 1986, Jungck [3] proved common fixed point theorem by introducing the notion of compatible mappings. This concept was frequently used to prove the existence theorems in common fixed points of noncompatible mappings and is also very interesting. Work along these lines has recently been initiated by Pant [6, 7]. Section 2 is devoted to definitions and known results which make the paper self reliant. In Section 3 we have proved a common fixed point theorem for sequence of mappings that generalizes the Theorem 2.8 of Aamri [1].

II. Preliminaries

Before proving our results, we need the following definitions and known results in this sequel.

Definition 2.1 ([3]). Let T and S be two self mappings of a metric space (X, d). T and S are said to be compatible if \( \lim_{n \to \infty} d(STx_n, TSx_n) = 0 \) whenever \( \{x_n\} \) is a sequence on X such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

Remark 2.2. Two weakly commuting maps are compatible, but the converse is not true as shown in [3].

Definition 2.3 ([3]). Two self mappings T and S of a metric space X are said to be weakly compatible if \( T = S \) for some \( u \in X \), then \( STu = TSu \).

Note 2.4. Two compatible maps are weakly compatible.

M. Aamri [1] introduced the concept (E.A) in the following way.

Definition 2.5 (Aamri [1]). Let S and T be two self mappings of a metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence \( \{x_n\} \) such that \( \lim_{n \to \infty} Sx_n = \lim_{n \to \infty} Tx_n = t \) for some \( t \in X \).

Definition 2.6 (Aamri [1]). Two self mappings S and T of a metric space (X, d) will be non compatible if there exists at least one sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} d(STx_n, TSx_n) \) is either nonzero or non-existent.

Remark 2.7. Two non compatible self mappings of a metric space (X, d) satisfy the property (E.A).

Aamri [1] proved the following theorems.

Theorem 2.8. Let S and T be two weakly compatible self mappings of a metric space (X, d) such that (i) T and S satisfy the property (E.A), (ii) \( d(Tx, Tx) < \frac{1}{2} \max\{d(Sx, Sx), d(Ty, Ty)\} \), (iii) \( TX \subseteq X \). Then T and S have a unique common fixed point.

III. Main Results

In this section we prove common fixed point theorem for sequence of mappings that generalizes Theorem 2.8.

Theorem 3.1 Suppose that \( \{A_i\}, \{T_i\} \) be two weakly compatible self mappings of a metric space (X, d) such that (1) For every i, \( A_i \subseteq TX \) and \( T_i \) satisfies the property (E.A). (2) \( d(A_i, A_j) < \frac{1}{2} \max\{d(Tx, Ty), d(A_i, T_i) + d(A_j, T_j)\} \) for \( x \neq y \) and for every i. If \( T_i, S, X \) is a complete subspace of X, then \( A_i \) and \( T_i \) have a unique common fixed point.

Proof: Suppose that \( A_i \) and \( T_i \) satisfies the property (E.A) there exists in X a sequence \( (x_n) \) satisfying \( \lim_{n \to \infty} A_i, x_n = \lim_{n \to \infty} T_i, x_n = t \) for some \( t \in X \) for every i.

Suppose that \( T_i \) is complete. Then \( \lim_{n \to \infty} T_i, x_n = T_i, a \) for some a \( \in X \).

Also \( \lim_{n \to \infty} A_i, x_n = T_i, a \). We show that \( A_i, a = T_i, a \).

Suppose that \( A_i, a \neq T_i, a \). Condition (3) implies \( d(A_i, a) < \frac{1}{2} \max\{d(TX, Ta), d(A_i, T_i) + d(A_i, Ta)\} \), \( A_i, a = T_i, a \).
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Letting \( n \to +\infty \) yields,
\[
\begin{align*}
&d(T_\infty, A_\infty) = \max\{d(T_\infty, A_\infty), d(T_\infty, A_\infty) + d(A_\infty, T_\infty)\}/2, \\
&\lim_{n \to +\infty} d(T_n, A_n) = \max\{d(T_n, A_n), d(A_n, T_n)\}/2. \\
\end{align*}
\]

For sufficiently large \( n \), \( T_n \) and \( A_n \) are weakly compatible. Then we have,
\[
\begin{align*}
&d(A_n, A_n, A_n) \leq \max\{d(T_n, A_n), d(A_n, T_n) + d(A_n, A_n)\}/2, \\
&\lim_{n \to +\infty} d(A_n, T_n) \leq \max\{d(A_n, A_n), d(A_n, T_n)\}/2. \\
\end{align*}
\]

Define \( A_n \) and \( T_n \) satisfy the property (E.A) for the sequence \( x_n = 2 + 2n \), \( n = 1, 2, \ldots \), \( n \to +\infty \).

Then (1) \( A_n \) and \( T_n \) satisfy the property (E.A) for the sequence \( x_n = 2 + 2n, n = 1, 2, \ldots \).

(2) \( T_n \) and \( A_n \) satisfy the property (E.A).

(3) \( A_n \) or \( T_n \) satisfy the property (E.A).

(4) \( A_n, T_n \) satisfy the property (E.A).

Theorem 3.2. Let \( A_n, B_n, T_n, S_n \) be self maps of a metric space \((X,d)\) such that

(1) \( A_n \subset T_n \) and \( B_n \subset S_n \) for every \( i \).

(2) \( A_n, S_n \) satisfy the property (E.A). (3) \( A_n \) or \( T_n \) satisfy the property (E.A).

If the range of the one of the mappings \( A_n, B_n, S_n \) or \( T_n \) is a complete subspace of \( X \),

Then (I) \( A_n \) and \( S_n \) have a common fixed point. (II) \( B_n \) or \( T_n \) have a common fixed point provided that \( B_n, T_n \) have a unique common fixed point for all \( i \).

Example for theorem 3.1: Let \( X = [1, \infty) \) with the usual metric \( d(x, y) = |x - y| \).

Define \( A_n, T_n : X \to X \) by \( A_n = 3x - 1 \) and \( T_n : x \to 1 + x \) \( \forall x \in X \) and \( i \in \mathbb{N} \).

Then (1) \( A_n \) and \( T_n \) satisfy the property (E.A) for the sequence \( x_n = 2 + 2n, n = 1, 2, \ldots \).

To prove, for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that \( |x - y| < \delta \Rightarrow |A_n x - A_n y| < \varepsilon \).

(2) \( T_n \) and \( A_n \) satisfy the property (E.A).

(3) \( A_n \) or \( T_n \) satisfy the property (E.A).

(4) \( A_n, T_n \) satisfy the property (E.A).
The following result due to U. Karuppiah [2] is a special case of the previous theorem 3.2.

Theorem 3.2: Let \( \{A_i\} \), S and T be self maps of a metric space \((X, d)\) such that

1. \( A_iX \subset TX \) and \( A_iX \subset SX \) for every i.

2. \( (A_i, S) \) is weakly compatible.

The proof is similar when \( T \subseteq X \).
(3) \((A_1, S)\) or \((A_i, T)\), \(i > 1\) satisfies the property (E.A).

(4) \(d(A_1x, A_iy) < \max\{d(Sx, Ty), d(A_1x, Sx), d(A_iy, Ty), d(A_1x, Ty), d(A_iy, Sx)\}\) for \(i > 1\).

If the range of the one of the mappings \(\{A_i\}\), \(S\) or \(T\) is a complete subspace of \(X\), then

(I) \(A_1\) and \(S\) have a common fixed point.

(II) \(A_i, i > 1\) and \(T\) have a common fixed point provided that \((A_k, T)\) for some \(k > 1\) is weakly compatible.

(III) \(A_i, S\) and \(T\) have a unique common fixed point provided that (I) and (II) are true.

References

[2] U. Karuppiah and M. Marudai, Common fixed point theorems for sequence of mappings under strict contractive conditions,