A Redefined Riemann Zeta Function Represented via Functional Equations Using the Osborne’s Rule

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Abstract: Intrigues most researchers about the Riemann zeta hypothesis is the ability to employ cum different approaches with instinctive mindset to obtain some very interesting results. Motivated by their style of reasoning, the result obtained in this work of redefining or re-representation of Riemann zeta function in different forms by employing different techniques on two functional equations made the results better, simpler and concise new representations of Riemann zeta function.

Keywords: Analytic Continuation, Osborne’s rule, Riemann Zeta Function

THE OSBORNE’S RULE APPROACH

Osborne’s rule gave a more comprehensive method for hyperbolic functions in terms of the Euler method.

\( \text{recall from coshx} = e^x + e^{-x} \quad \text{and the fact that} \quad e^{i\theta} = \cos \theta + isin \theta \)

Imposing that \( \Re(e^s) = \cosh x \) (1.0)

\[ \Re \left( e^{\frac{s^2}{2}} \right) = \cosh \left( \frac{s^2}{2} \right) \] (1.1)

From the Riemann zeta function

\[ \zeta(s) = \frac{4}{s(s-1)(4)} \sum_{n=1}^{\infty} \frac{e^{-n\pi x}}{(nn)^2} \left( (nn)^2 \pi^2 \chi^2 - \frac{3}{2} nn \pi x^2 \right) \cosh \left( \frac{x}{2} \right) \text{ln}(x) dx \] (1.2)

Substituting (1.0), into equation (1.1), then the equation (1.2) becomes

\[ \zeta(s) = \frac{4}{s(s-1)(4)} \sum_{n=1}^{\infty} \frac{e^{-n\pi x}}{(nn)^2} \left( (nn)^2 \pi^2 \chi^2 - \frac{3}{2} nn \pi x^2 \right) e^{\frac{x^2}{2}} \text{ln}(x) dx \] (1.3)

considering the right arm of the r. h. s side of the equation (1.3)

\[ \int_{1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{-n\pi x}}{(nn)^2} \left( (nn)^2 \pi^2 \chi^2 - \frac{3}{2} nn \pi x^2 \right) e^{\frac{x^2}{2}} \text{ln}(x) dx \]

= \[ \int_{1}^{\infty} \sum_{n=1}^{\infty} \left( (nn)^2 \pi^2 \chi^2 - \frac{3}{2} nn \pi x^2 \right) e^{-n\pi x + \frac{x^2}{2}} \text{ln}(x) dx \] (1.4)

= \[ \sum_{n=1}^{\infty} \left( \int_{1}^{\infty} (nn)^2 \pi^2 \chi^2 e^{-n\pi x + \frac{x^2}{2}} \text{ln}(x) dx - \frac{3}{2} nn \pi x^2 e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \right) \] (1.5)

considering the left hand side of the equation (1.6) i.e

\[ \sum_{n=1}^{\infty} (nn)^2 \pi^2 \int_{1}^{\infty} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \] (1.7)

Let \( I = \int_{1}^{\infty} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \) and introducing the integration by part:

then \( I = \left( 14 \times 10^{+4} \right) \int_{1}^{\infty} \frac{2}{9} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \) (1.8)

From equation (1.8)

Suppose \( M_1 = \int_{1}^{\infty} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \), by furthering the integration by part from (1.9),

\[ M_1 = \frac{4}{9} (nn)^3 \pi^3 \left( \frac{4}{13} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx - \frac{4}{3} nn \pi x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) dx \right) \] (2.0)

Substituting (2.0) into (1.8), we obtain

\[ I = \left( \frac{10}{18s+4a} \right) \frac{2}{9} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) - \frac{4}{9} (nn)^3 \pi^3 \left( M + \frac{4}{26} M \right) = \] (2.1)

From (2.0), we have \( M_1 = \left( \frac{26}{26+4a} \right) \left( \frac{4}{13} x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) + \frac{4}{13} nn \pi x^n e^{-nn \pi x + \frac{x^2}{2}} \text{ln}(x) \right) \)
substituting 2.0 into 1.8
\[ I = \left( \frac{18(nn)^2\pi^2}{18+4s} \right) \frac{4}{9} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} \left( \frac{4(nn)^2\pi^2}{18+4s} \right) \left( \frac{26}{18+4s} \right) \left[ \frac{13}{18+4s} e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} + \frac{4}{13} e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} \right] dx \]
\[ = \left( \frac{18(nn)^2\pi^2}{18+4s} \right) \frac{4}{9} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} - \frac{468(nn)^2\pi^2}{1053+396s+36s^2} \left( \frac{4}{13} \right) e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} - \frac{4}{13} e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} \right] dx \]
\[ = - \frac{144}{1053+234s} (nn)^4 \pi^4 \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \]
\[ (2.1) \]

considering the right hand side of the equation (1.6) i.e.
\[ = \sum_{n=1}^{\infty} 2 \pi \sum_{n=1}^{\infty} \frac{1}{2} e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx = \frac{3}{2} \sum_{n=1}^{\infty} \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \]
\[ (2.3) \]

Suppose \( M_2 = \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \)
\[ (2.4) \]

applying integration by part on equation (2.4).
\[ M_2 = \left( \frac{5}{5+2s} \right) \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx + 16\pi \int_1^{\infty} x^3 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx + 16\pi \int_1^{\infty} x^2 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx - 16\pi \int_1^{\infty} x e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \]
\[ \]
\[ (2.5) \]

recall \( M_2 = \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \) then from (1.6).
\[ M_2 = \left( \frac{2}{3nn} \right) \int_1^{\infty} x^4 e^{-\text{ln}x + \frac{1}{2}\text{ln}\pi} dx \]
\[ (2.6) \]

from the equation (1.6), substituting \( l \) and \( M_2 \), to obtain
\[ \]
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Let \( \sum_{n=1}^{\infty} e^{-\pi n x} = \psi(x) \), and \( e^{\ln x} = x \).

Then equations 3.1 becomes

\[
\psi(x)x^2 \left[ \frac{(18)\pi^2}{18+4s} \right] \frac{4}{9} x^4 - \frac{468(n\pi)^2}{1053+396s+36s^2} \left( \frac{4}{13} x^3 \right) - \frac{40x^2}{n\pi (75+30s)} + \frac{160n\pi x^3}{n\pi (675+3270s} \right] - \frac{144}{1053+234s} \left( n\pi \right)^4 x^4 \int_0^\infty e^{-\pi n x} + \frac{5}{2} \ln dx = \frac{16(n\pi)^2}{n\pi (675+270s)} M_1 + \frac{16n\pi x}{n\pi (1375+540s)} I \quad (3.2)
\]

Substituting (3.0) into (1.2)

\[
\zeta(s) = \frac{4}{2\pi^2 \zeta(2)} \left[ \psi(x)x^2 \left( \frac{(18)\pi^2}{18+4s} \right) \frac{4}{9} x^4 - \frac{468(n\pi)^2}{1053+396s+36s^2} \left( \frac{4}{13} x^3 \right) - \frac{40x^2}{n\pi (75+30s)} + \frac{160n\pi x^3}{n\pi (675+3270s} \right] - \frac{1441053+234s(n\pi)^4 x^{134}e^{-\pi n x} + s2\ln dx - 16n\pi M_1 + 16n\pi x M_1 + 16n\pi x M_1} {2\pi^2 \zeta(2)} \quad (3.3)
\]

from the above representation the \( \zeta(s) \) is valid for \( s > 1 \)

Conclusion

This representation really shows how the complex part of the equation could be eliminated by introducing a hyperbolic function from Enoch and Adeyeye(2012) result which was the reason why employ the technique of the Osborne’s Rule approach. Also from the result obtained, by observation one see how much primes numbers are been played around the synthetic process. This actually obeys the rules guiding the Number theory (every integer is a product of prime). This method helps in removing the “i” in the result using the Euler’s Method though we could not see how the integers are been expressed in the products of primes much but this validate some of the claims of researchers like Euler (1748), Odgers (2004). Thus, the Osborne’s approach is more preferable.

References