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On the Differentiability of Multivariable Functions

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Abstract : *Objective of this article is to review the differentiability of functions of several real variables.*
Keywords: *Differentiability, Linear Transformation, Multivariable Function.*

I. INTRODUCTION

Calculus is unarguably one of the most fascinating and useful subject of the modern science and engineering curriculum. Calculus of functions of several variables, also called multivariable calculus. The generalization of calculus from the real line $\mathbf{R}$ to n-space $\mathbf{R}^n$ makes the study of calculus extremely interesting. A further generalization of calculus to spaces more general than $\mathbf{R}^n$, called calculus on manifolds. In the study of calculus the differentiability of functions is of prime importance since derivatives appear almost everywhere in the field of science and engineering. Applications of derivative in many physical problems e.g. approximation of functions about a point, error analysis, obtaining extremum, population growth etc. are of great practical interest.

While teaching calculus to undergraduate students I observed that students grasp the idea of differentiability in one variable case quite easily, nevertheless they were not at the same ease with the differentiability concept of multivariable functions. In this article an attempt has been made to exposit differentiability of multivariable functions in an elegant manner to address pedagogical problems. A reader having knowledge of basic calculus and linear algebra will find this article fairly accessible.

II. DIFFERENTIABILITY OF SINGLE VARIABLE FUNCTIONS

The idea of the notion of the derivative originated from a problem in geometry – the problem of finding the tangent at a point of a curve. Though derivative was originally formulated to study the problem of tangents, sooner it was observed that it also provides a way to calculate velocity and, more generally the rate of change of a function [1].

To start with the basic definition of derivative first we wish to fix our notation. Most of the texts of calculus describe $\frac{dy}{dx}$ as another notation for $f'(x)$ where $y = f(x)$, but the fact that $\frac{dy}{dx}$ and $f'(x)$ are not interchangeable is evident when you consider that one does not write $\frac{dy}{d4}$ for $f'(4)$. The obvious objection for this is that it does not make any sense to differentiate with respect to a constant i.e. we cannot have a rate of change with respect to something that is not changing at all. Though, if there is no risk of confusion one can use the notations $\frac{dy}{dx}$ and $f'(x)$ freely and interchangeably [2].

Now we recall the definition of differentiability of a real valued function of a real variable [3].

Definition: Let A be a subset of $\mathbf{R}$ containing a neighborhood of the point a, that is a is an interior point of A. The derivative of a function $f : \mathrm{A} \to \mathbf{R}$ at a, defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists. In this case, we say that f is differentiable at point a.

The following facts are immediate consequence of the above definition:

(i) Differentiable functions are continuous.

At this juncture it is worthwhile to mention that above consequence merely tells us that if a function is differentiable at a point of the domain then it must be continuous there but fails to provide further information about the continuity of the derivative and existence of subsequent derivatives. For example the function $f(x) = x^2 \sin\left(\frac{1}{x}\right) = 0,\ x \neq 0$ and $f(0) = 0$ has derivative $f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$, $x \neq 0$ and $f'(0) = 0$, here $\lim_{x\to 0} 2x\sin\left(\frac{1}{x}\right) = 0$ but $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not have a definite value, so f' is not continuous at 0.

One can check that the function $g(x) = x|x|$ is differentiable for all x in $\mathbf{R}$, with $g'(x) = |x|$ but derivative $g'(x) = |x|$ is no more differentiable at 0.

Remark: Regarding the existence of subsequent derivatives in a neighborhood of the domain, behaviour of complex valued functions is pretty stronger than real ones. In case of the complex-valued function of a complex variable, existence of first derivative in a neighborhood of the domain guarantees not only the continuity of function but the existence of derivatives of all orders.

(ii) Composites of differentiable functions are differentiable. (Chain rule)

III. DIFFERENTIABILITY OF MULTIVARIABLE FUNCTIONS

We seek now to define the derivative of multivariable functions. Suppose $D \subseteq \mathbf{R}^n$ containing a neighborhood of point $\mathbf{a}$ (i.e. $\mathbf{a} = (a_1, \ldots, a_n)$ is an interior point of D) and a function $f : D \to \mathbf{R}$. One may tempt to define the differentiability of a multivariable function just by replacing a and h in definition of the derivative of single variable function by points of $\mathbf{R}^n$ but this does not make any sense, as division of a real number by a point in $\mathbf{R}^n$ has not been defined if $n > 1$.

Here goes another attempt at a definition:

Definition: Let $D \subseteq \mathbf{R}^n$ containing a neighborhood of point $\mathbf{a}$ (i.e. $\mathbf{a}$ is an interior point of D). Let a function $f : D \to \mathbf{R}$. Given a unit vector $\mathbf{u} \in \mathbf{R}^n$ with $\mathbf{u} \neq 0$ suppose $t \in \mathbf{R}$ such that $\mathbf{a} + t\mathbf{u} \in D$, define

$$\mathrm{D}_{\mathbf{u}} f(\mathbf{a}) = \lim_{t\to 0} \frac{f(\mathbf{a} + t\mathbf{u}) - f(\mathbf{a})}{t}$$

provided the limit exists. This limit depends both on point $\mathbf{a}$ and on vector $\mathbf{u}$; it is called the directional derivative of f at $\mathbf{a}$ along the vector $\mathbf{u}$ or the directional derivative of f at $\mathbf{a}$ with respect to vector $\mathbf{u}$. Some authors denote directional derivative by $f'(\mathbf{a};\mathbf{u})$ instead $\mathrm{D}_{\mathbf{u}} f(\mathbf{a})$.

Example: Let $f : \mathbf{R}^2 \to \mathbf{R}$, defined by $f(x_1, x_2) = \dfrac{x_1^2 x_2}{x_1^4 + x_2^2}$, $(x_1, x_2) \neq \mathbf{0}$ with $f(\mathbf{0}) = 0$.

It can be easily checked that all directional derivatives of f exist at $\mathbf{0} = (0,0)$.

Let $\mathbf{0} \neq \mathbf{u} = (u_1, u_2) \in \mathbf{R}^2$. A straightforward calculation yields

$$\mathrm{D}_{\mathbf{u}} f(\mathbf{0}) = \begin{cases} \dfrac{u_1^2}{u_2} & \text{if } u_2 \neq 0 \\ 0 & \text{if } u_2 = 0 \end{cases}$$

i.e. function has all directional derivatives at $\mathbf{0}$. It can be verified that as we approach to origin (e.g. along the curve $x_2 = kx_1^2,\ k \neq 0$) limit of the given function does not exist; hence we infer that function is not continuous at origin however it has all directional derivatives at origin.

It appears that the "directional derivative" is an appropriate generalization of the notion of "derivative" but this is not true, because differentiability implies continuity and from above example it is clear that function possesses all directional derivatives at origin but not continuous at origin. Obviously directional derivative is very useful definition but it restricts us to study the change of f in one direction at a time and in particular, if we choose $\{e_1, \ldots, e_n\}$ as standard basis of the n-space $\mathbf{R}^n$, for $1 \leq i \leq n$ the directional derivative at $\mathbf{a}$ in the direction of $\mathbf{u} = e_i$ is merely partial derivative of f in the i^{th} coordinate direction, denoted by $\mathrm{D}_i f(\mathbf{a})$ or

$\frac{\partial f}{\partial x_i}(\mathbf{a})$. For instance in above example taking $\{e_1 = (1,0),\ e_2 = (0,1)\}$ as standard basis of $\mathbf{R}^2$ gives $\mathrm{D}_1 f(\mathbf{0}) = 0$ and $\mathrm{D}_2 f(\mathbf{0}) = 0$, respectively the first partial derivatives of f at origin, namely $\frac{\partial f}{\partial x_1}(\mathbf{0})$ and $\frac{\partial f}{\partial x_2}(\mathbf{0})$.

Now we seek something stronger to define the concept of differentiability of multivariable functions. For this purpose we revisit the differentiability definition of one variable case and take a closer look (geometrically). It can be observed that if a function is differentiable at an interior point of the domain then at that point there is a (local) tangent line associated with it i.e. differentiable functions are locally linear, or in other words differentiable functions enjoy reasonably good local approximation by linear functions. Intuitively this means, if a function f is differentiable at a point a then after sufficiently zooming-in around the point, graph of f locally looks like part of the straight line that is at the point of differentiability there does not exist any sharp corner (in addition it may be noted that at the point of differentiability local tangent line must not be vertical). For illustration if we observe the graph of function $f(x) = |x|$ (Fig. 1), at the origin no matter up to which magnitude we zoom-in around the point 0, locally it never looks like part of the straight line and a sharp corner at point 0 always remains there (inferring that function is not locally linear at the origin), hence function $f(x) = |x|$ is not differentiable at 0.

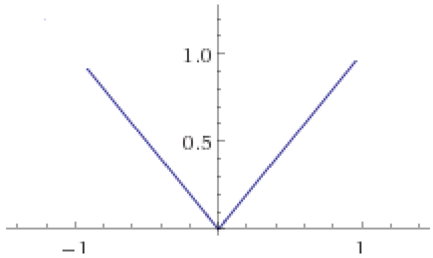

Figure 1 (*Graph is plotted using Wolfram Alpha Widgets*)

The geometrical interpretation of differentiability as existence of (local) tangent line in one variable case has an obvious generalization to higher dimensions. In higher dimension (i.e. for multivariable functions) there corresponds a (local) tangent plane at the point of differentiability. Taking a clue from this observation the definition of differentiability of single variable functions can be reformulated, so that it can be generalized to multivariable functions.

Definition: Let $D \subseteq \mathbf{R}$ containing a neighborhood of point a (i.e. a is an interior point of D). A function $f : D \to \mathbf{R}$ is said to be differentiable at $a \in D$ if there exist a number λ such that

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - \lambda h}{h} = 0$$

the number λ is called derivative of the function at a, that is $\lambda = f'(a)$. If we put $L(h) = f'(a)h$, obviously $L : \mathbf{R} \to \mathbf{R}$ is a linear transformation.

The above definition can be generalized for multivariable functions as follows:

Definition: Let $D \subseteq \mathbf{R}^n$ containing a neighborhood of point $\mathbf{a}$ (i.e. $\mathbf{a}$ is an interior point of D). A function $f : D \to \mathbf{R}$ is said to be differentiable at $\mathbf{a} \in D$ if there exist a linear transformation $\mathbf{L} : \mathbf{R}^n \to \mathbf{R}$, such that

$$\lim_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{L}(\mathbf{h})|}{\|\mathbf{h}\|} = 0$$

Since $\mathbf{h}$ is a point of $\mathbf{R}^n$ and $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{L}(\mathbf{h})$ a point of $\mathbf{R}$, so the norm (in 1-dimension mod) signs are necessary. The linear transformation $\mathbf{L}$, which is unique, is called the derivative of f at $\mathbf{a}$.

Remark: From above definition it is clear that $f:\mathbf{R}^2 \to \mathbf{R}$ is differentiable at $(a,b) \in D \subseteq \mathbf{R}^2$, if

$$\lim_{(h,k)\to(0,0)} \frac{|f(a+h,b+k)-f(a,b)-\mathbf{L}(h,k)|}{\|(h,k)\|} = 0 \ ,$$

The linear transformation $\mathbf{L}$ is given by $\mathbf{L}(h,k) = \alpha_1 h + \alpha_2 k$, $\alpha_1 = f_{x_1}(a,b)$ and $\alpha_2 = f_{x_2}(a,b)$.

Example [4]: Consider $f:\mathbf{R}^2 \to \mathbf{R}$ defined by $f(x_1,x_2) = |x_1 \, x_2|$, and investigate the differentiability of function at $(0,0)$.

An easy computation yields $f_{x_1}(0,0) = 0 = \alpha_1$ and $f_{x_2}(0,0) = 0 = \alpha_2$

Let $(0,0) \neq (h,k) \in \mathbf{R}^2$, then

$$\lim_{(h,k)\to(0,0)} \frac{|f(0+h,0+k)-f(0,0)-L(h,k)|}{\|(h,k)\|} = \lim_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-(0\,h+0\,k)|}{\sqrt{h^2+k^2}}$$

$$= \lim_{(h,k)\to(0,0)} \frac{|h\,k|}{\sqrt{h^2+k^2}} = 0$$

Since $|h| \le \sqrt{h^2+k^2}$, we have $\frac{|h\,k|}{\sqrt{h^2+k^2}} \le |k| \to 0$ as $(\mathrm{h},\mathrm{k}) \to (0,0)$.

Showing that $f(x_1,x_2) = |x_1 \, x_2|$ is differentiable at origin.

Analogous to differentiability of real valued functions of several real variables the differentiability of functions $f:\mathbf{R}^n \to \mathbf{R}^m$ can be defined [5].

Definition: Let $D \subseteq \mathbf{R}^n$ containing a neighborhood of point $\mathbf{a}$ (i.e. $\mathbf{a}$ is an interior point of D). A function $f:D \to \mathbf{R}^m$ is said to be differentiable at $\mathbf{a} \in D$ if there exist a linear transformation $\mathbf{L}:\mathbf{R}^n \to \mathbf{R}^m$, such that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|f(\mathbf{a}+\mathbf{h})-f(\mathbf{a})-\mathbf{L}(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

The linear transformation $\mathbf{L}$, which is unique, is called the derivative of f at point $\mathbf{a}$, denoted by $f'(\mathbf{a})$. Every linear transformation $\mathbf{L}:\mathbf{R}^n \to \mathbf{R}^m$ (with respect to standard bases of $\mathbf{R}^n$ and $\mathbf{R}^m$) can be given in terms of a $m \times n$ matrix, known as derivative matrix or Jacobian matrix. In particular if $f:\mathbf{R} \to \mathbf{R}$, $f'(a)$ is a 1×1 matrix, say $[\alpha]$ that is just a real number, and if $f:\mathbf{R}^2 \to \mathbf{R}$, derivative matrix of $f'(\mathbf{a})$ is given by a 1×2 matrix, say $[\alpha_1 \quad \alpha_2]$.

The Jacobian matrix of a transformation $\mathbf{F}:\mathbf{R}^n \to \mathbf{R}^m$, $(x_1,\ldots,x_n) \mapsto \mathbf{F}(F_1,\ldots,F_m)$, is an $m\times n$ matrix of the first partial derivatives of coordinate functions $F_i:\mathbf{R}^n \to \mathbf{R}$, $1 \le i \le m$ with respect to x_j, $1 \le j \le n$, given by

$$D\mathbf{F} = \left[\frac{\partial F_i}{\partial x_j}\right]_{m\times n}$$

That is if $f:D \to \mathbf{R}^m$ is differentiable at an interior point $\mathbf{a} \in D \subseteq \mathbf{R}^n$ then Jacobian matrix of f at $\mathbf{a}$ exists and given by a $m\times n$ matrix $Df(\mathbf{a}) = \left[\frac{\partial f_i}{\partial x_j}(\mathbf{a})\right]_{m\times n}$, which is unique. It is important to note that differentiability of f at $\mathbf{a}$ implies existence of all first partial derivatives of f at $\mathbf{a}$ (even all directional derivatives of f at $\mathbf{a}$ exist) but the converse need not be true. Though existence of continuous first partial derivatives of f throughout some neighbourhood of $\mathbf{a}$ guarantees differentiability of f at $\mathbf{a}$ but this is only sufficient condition and not necessary. For example the following function is differentiable at the origin but its first partial derivatives are not continuous at the origin:

Let $f:\mathbf{R}^2 \to \mathbf{R}$ be defined by

$$f(x_1,x_2)=\left(x_1^2+x_2^2\right)\sin\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right),\ (x_1,x_2)\neq\mathbf{0} \text{ and } f(x_1,x_2)=0 \text{ if } (x_1,x_2)=\mathbf{0}.$$

We have, $\dfrac{\partial f}{\partial x_1}(0,0)=\lim\limits_{h\to 0}\dfrac{f(0+h,0)-f(0,0)}{h}=\lim\limits_{h\to 0}\dfrac{h^2\sin\left(\frac{1}{|h|}\right)-0}{h}=0=\alpha_1$ and

Similarly $\dfrac{\partial f}{\partial x_2}(0,0)=0=\alpha_2$

Let $(0,0)\neq(h,k)\in\mathbf{R}^2$, then

$$\lim_{(h,k)\to(0,0)}\frac{|f(0+h,0+k)-f(0,0)-L(h,k)|}{\|(h,k)\|}=\lim_{(h,k)\to(0,0)}\frac{|f(h,k)-f(0,0)-(0h+0k)|}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0)}\frac{\left|(h^2+k^2)\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)-0\right|}{\sqrt{h^2+k^2}}$$

$$=\lim_{(h,k)\to(0,0}\left(\sqrt{h^2+k^2}\right)\left|\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)\right|$$

Since function sine is bounded by both -1 and 1, therefore $\left|\sin\left(\frac{1}{\sqrt{h^2+k^2}}\right)\right|\leq 1$, hence above limit becomes

$\leq\lim\limits_{(h,k)\to(0,0}\left(\sqrt{h^2+k^2}\right)$, which is equal to 0, showing that function is differentiable at origin.

We have already seen that both first partial derivatives of the function at the origin are 0. Now other than the origin

$$\frac{\partial f}{\partial x_1}(x_1,x_2)=2x_1\sin\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right)-\frac{x_1}{\sqrt{x_1^2+x_2^2}}\cos\left(\frac{1}{\sqrt{x_1^2+x_2^2}}\right)$$

Consider above partial derivative along x_1-axis (that is along $x_2=0$), then

$$\frac{\partial f}{\partial x_1}(x_1,0)=2x_1\sin\left(\frac{1}{|x_1|}\right)-\frac{x_1}{|x_1|}\cos\left(\frac{1}{|x_1|}\right)$$

Now if $x_1\to 0$, the first term $2x_1\sin\left(\frac{1}{|x_1|}\right)\to 0$ but the second term $\frac{x_1}{|x_1|}\cos\left(\frac{1}{|x_1|}\right)$ oscillates between -1 and 1 (since for $x_1\neq 0,\ \frac{x_1}{|x_1|}=-1 \text{ or } 1$ depending upon the sign of x_1, and cosine function is bounded by -1 and 1), hence this limit does not exist thereby showing that $\frac{\partial f}{\partial x_1}$ is not continuous at the origin (though $\frac{\partial f}{\partial x_1}(0,0)$ exists and is equal to 0).

Similarly it can be shown that $\frac{\partial f}{\partial x_2}$ not continuous at the origin (though $\frac{\partial f}{\partial x_2}(0,0)$ exists and is equal to 0).

IV. CONCLUSION

The concept of differentiability of functions starting from single real variable function to several real variables functions has been reviewed. It is hoped that article will benefit the readers to develop a better understanding of the differentiability concept in higher dimensions.

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