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Projectively Flat Finsler Space With Special (α, β)-Metrics

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I. Introduction.

In 1990 M. Matsumoto has considered the projectively flatness of Finsler spaces with (α, β)-metric [5]. In particular, the Randers space, the Kropina space and the special generalized Kropina space are considered in detail. Here α is Riemannian metric and β is a differential one-form.

In the present chapter we consider the projective flatness of Finsler space with special (α, β)-metrics. In particular, Matsumoto metric $\alpha^2/(\alpha - \beta)$, special generalized Matsumoto metric $\beta^2/(\beta - \alpha)$ and the metric $\alpha + (\beta^2/\alpha)$ are considered in detail.

II. Projective flatness of (α, β)-metric.

Consider a Finsler space with (α, β)-metric L(α, β), where L is fundamental function positively homogeneous of degree one in α and β, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is Riemannian metric and $\beta = b_i(x)\, y^i$ is one-form.

Firstly, we are concerned with the associated Riemannian space with metric α and define

(1.1) (a) $2r_{ij} = b_{i;j} + b_{j;i} = \partial_j b_i + \partial_i b_j + 2\gamma^s_{ij} b_s$,

(b) $2s_{ij} = b_{i;j} - b_{j;i} = \partial_j b_i - \partial_i b_j$

Which are symmetric and skew symmetric tensors of order 2. Here (;) denote the covariant differentiation with respect to Riemannian Christoffel symbols $\gamma^i_{jk}(x)$. Further, we define

(1.2) $s^i_j = a^{ir} s_{rj}$, $b^i = a^{ir} b_r$, $s_i = b^r s_{ri}$, $b^2 = a^{rs} b_r b_s$.

Here a^{ij} are conjugate metric tensor of a_{ij}.

Next, we consider the Berwald connection $B\Gamma = (G^i_{jk}, G^i_j, 0)$ of the Finsler space with the (α, β)-metric L(α, β). As, is well known, we have

$$G^i_{jk} = \dot{\partial}_k G^i_j,\ G^i_j = \dot{\partial}_j G^i,$$

$$2G_j = g_{ij} G^i = y^r \dot{\partial}_j \partial_r F - \partial_j F, \qquad F = L^2/2$$

where g^{ij} denote the conjugate metric tensor of metric $g_{ij}(x, y)$ of the Finsler space.

If we put

(1.3) $$2B^i = 2G^i - \gamma^i_{00},$$

where the subscript 0 denote the contraction by y^i i.e. $\gamma^i_{00} = \gamma^i_{jk} y^j y^k$, then the equation (1.1) of [5] gives

(1.4) $$B^i = \left(\frac{E}{\alpha}\right) y^i + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 - \left(\frac{\alpha L\alpha\alpha}{L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right)\left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}\right),$$

where E and C satisfy

(1.5) (a) $C + \left(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\right) s_0 + \left(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\right)(\alpha^2 b^2 - \beta^2)\left(C + \frac{\alpha r_{00}}{2\beta}\right) = 0,$

(b) $\left(\frac{2L}{\alpha}\right) E = \left(\frac{2\beta L_\beta}{\alpha}\right) C + L_\beta r_{00}.$

Here $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_\beta = \frac{\partial L}{\partial \beta}$, $L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$ and so on.

Now, we consider projectively flatness of a Finsler space. A Finsler space is projectively flat if and only if the space is covered by rectilinear coordinate neighbourhoods i.e. in these neighbourhoods geodesics can be represented by (n−1) linear equations of the coordinates. Therefore G^i is proportional to y^i ([1], [6]). Thus there exist a function P(x, y) satisfying $G^i = P y^i$. Hence from (1.3) and (1.4), we get

(1.6) $$\frac{1}{2}\gamma^i_{00} + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 + \left(\frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right) b^i = p y^i,$$

where $p = P - \frac{E}{\alpha} + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right)$. Contracting (1.6) by $y_i = a_{ir} y^r$ and using $s^i_o y_i = 0$, $b^i y_i = \beta$, we get

(1.7) $$\frac{1}{2}\gamma_{000} + \left(\frac{\alpha^2 L_{\alpha\alpha}}{L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right) = p\alpha^2.$$

Now, eliminating p from (1.6) and (1.7), we get

(1.8) $$\frac{1}{2}\left(\gamma^i_{00} - \frac{\gamma_{000} y^i}{\alpha^2}\right) + \left(\frac{\alpha L_\beta}{L_\alpha}\right) s^i_0 + \left(\frac{L_{\alpha\alpha}}{L_\alpha}\right)\left(C + \frac{\alpha r_{00}}{2\beta}\right)\left(\frac{\alpha^2 b^i}{\beta} - y^i\right) = 0.$$

Thus we have the following [5] "A Finsler space with an (α, β)- metric L(α, β) is projectively flat if and only if the space is covered by coordinate neighbourhoods in which equation (1.8) is satisfied,"

III. Projectively flat Matsumoto space

We consider the Finsler space F with Matsumoto metric $L(\alpha, \beta) = \alpha^2/(\alpha - \beta)$, then,

(2.1) $$L_\alpha = \frac{\alpha(\alpha - 2\beta)}{(\alpha-\beta)^2}, \qquad L_\beta = \frac{\alpha^2}{(\alpha-\beta)^2},$$

$$L_{\alpha\alpha} = \frac{2\beta^2}{(\alpha-\beta)^3}, \qquad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}.$$

From (2.1), the following identities hold.

(2.2) $\alpha(\alpha-\beta) L_\alpha = (\alpha - 2\beta) L$, $(\alpha-\beta) L_\beta = L$,
$\alpha^2 (\alpha-\beta)^2 L_{\alpha\alpha} = 2\beta^2 L$, $(\alpha-\beta)^2 L_{\beta\beta} = 2L$.

Now, equation (1.5) can be written as

(2.3) $$\left\{1 + \left(\frac{\alpha L_{\alpha\alpha}}{\beta^2 L_\alpha}\right)(\alpha^2 b^2 - \beta^2)\right\}\left(C + \frac{\alpha r_{00}}{2\beta}\right) = \frac{\alpha}{2\beta} r_{00} - \left(\frac{\alpha^2 L_\beta}{\beta L_\alpha}\right) s_0.$$

Using (2.1) in (2.3), we get

$$2\beta\{(1+2b^2)\beta-3\beta\}\left(C+\frac{\alpha r_{00}}{2\beta}\right)=(\alpha-\beta)\{(\alpha-2\beta)r_{00}-2\alpha^2 s_0\}.$$

Further, using this value in (1.8) with the help of (2.1), we get

$$\frac{1}{2}\left(\gamma^i_{00}-\frac{\gamma_{000}y^i}{\alpha^2}\right)+\left(\frac{\alpha^2}{\alpha-2\beta}\right)s^i_0+\frac{\beta\{(\alpha-2\beta)r_{00}-2\alpha^2 s_0\}}{\alpha\{(1+2b^2)\alpha-3\beta\}(\alpha-2\beta)}\left(\frac{\alpha^2 b^i}{\beta}-y^i\right)=0$$

which can also be re-arranged as fallows

(2.4) $\{-\beta(5+4b^2)(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)+2\alpha^4(1+2b^2)s^i_0$

$-4(\beta r_{00}+\alpha^2 s_0)(\alpha^2 b^i-\beta y^i)\}\alpha$

$+[\{(1+2b^2)\alpha^2+6\beta^2\}(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)$

$-6\alpha^4\beta s^i_0+2\alpha^2 r_{00}(\alpha^2 b^i-\beta y^i)]=0,$

which is of the form

(2.5) $P^i+\alpha Q^i=0,$

where

(2.6) $P^i=\{(1+2b^2)\alpha^2+6\beta^2\}(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)$

$-6\alpha^4\beta s^i_0+2\alpha^2 r_{00}(\alpha^2 b^i-\beta y^i)$

(2.7) $Q^i=-\beta(5+4b^2)(\alpha^2\gamma^i_{00}-\gamma_{000}y^i)+2\alpha^4(1+2b^2)s^i_0$

$-4(\beta r_{00}+\alpha^2 s_0)(\alpha^2 b^i-\beta y^i)\}.$

From (2.6) and (2.7) it is clear that both P^i and Q^i are rational functions in (x^i, y^j), but α is irrational function in (x^i, y^j). Therefore (2.5) holds good if and only if

(2.8) $P^i=0,\quad Q^i=0.$

Consider $P^i=0$ then it can be re-arranged such that $\beta^2\gamma_{000}y^i$ must have a factor α^2. Therefore there exist functions $\lambda_i(x)$ such that

(2.9) $\gamma_{000}=\alpha^2\lambda_0.$

Using (2.9) in $Q^i=0$, we observe that $\beta r_{00}y^i$ has a factor α^2. Therefore there exist a function $\mu(x)$ such that

(2.10) $r_{00}=\alpha^2\mu(x).$

Using (2.9) and (2.10) in $P^i=0$, we get

(2.11) $\{(1+2b^2)\alpha^2+6\beta^2\}(\gamma^i_{00}-\lambda_0 y^i)$

$-6\alpha^2\beta s^i_0+2\alpha^2\mu(\alpha^2 b^i-\beta y^i)=0.$

From (2.11), we observed that $\beta^2(\gamma^i_{00}-\lambda_0 y^i)$ has a factor α^2. Therefore there exist functions $\nu^i(x)$ such that $\gamma^i_{00}-\lambda_0 y^i=\alpha^2\nu^i(x)$. Contracting this by $y_i=a_{ir}y^r$ and using (2.9), we get $\nu^i(x)=0$, which gives

(2.12) $\gamma^i_{00}=\lambda_0 y^i$

Successive differentiation of (2.12) with respect to y^j and y^k gives

(2.13) $$\gamma^i_{jk} = \lambda_k \delta^i_j + \lambda_j \delta^i_k,$$

This equation shows the projective flatness of the associated Riemannian space of given Finsler space. Thus we have the following,

Theorem (2.1). If the Matsumoto space is projectively flat then its associated Riemannian space is also projectively flat.

Further, Using (2.12) in (2.11) we get

(2.14) $$\mu(\alpha^2 b^i - \beta y^i) = 3\beta s^i_0.$$

Contracting this by b_i and using $s^i_0 b_i = s_0$ we get

(2.15) $$\mu = \frac{3\beta s_0}{\alpha^2 b^2 - \beta^2}.$$

Putting the value of μ from (2.15) in (2.14) and (2.10), we get

(2.16) $$s^i_0 = \frac{s_0(\alpha^2 b^i - \beta y^i)}{\alpha^2 b^2 - \beta^2}, \qquad r_{00} = \frac{3\alpha^2 \beta s_0}{\alpha^2 b^2 - \beta^2}.$$

Using (2.9), (2.12) and (2.16) in $P^i = 0$, we get

$$\frac{2\alpha^2(\alpha^2 - 4\beta^2)s_0}{\alpha^2 b^2 - \beta^2}(\alpha^2 b^i - \beta y^i) = 0.$$

Contracting this by b_i, we get $s_0 = 0$, therefore from (2.16) we get $s^i_0 = 0$, $r_{00} = 0$, which gives $s_{ij} = 0$, $r_{ij} = 0$. Hence from (1.1), we get $b_{i;j} = 0$. Therefore if the Matsumoto space is projectively flat, then $b_{i;j} = 0$.

Conversely suppose that the associated Riemannian space of the Matsumoto space is projectively flat and $b_{i;j} = 0$. Then equation (2.12) is satisfied and $s_{ij} = 0$, $r_{ij} = 0$, which gives $r_{00} = 0$, $s^i_0 = 0$, $s_0 = 0$. Therefore using all these in equation (2.4), we see that this equation holds identically. Hence the Matsumoto space is projectively flat.

Summarizing above all , we have the following

Theorem (2.2). A Matsumoto space projectively flat if and only if its associated Riemannian space is projectively flat and $b_{i;j} = 0$.

IV. Projectively flat Finsler space with metric $\beta^2/(\beta - \alpha)$

We consider the Finsler space F with the metric

(3.1) $$L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha},$$

which is the metric obtained by interchanging α and β in Matsumoto metric and is special Matsumoto metric. From (3.1), we have

(3.2) $$L_\alpha = \frac{\beta^2}{(\beta-\alpha)^2}, \qquad L_\beta = \frac{\beta(\beta - 2\alpha)}{(\beta-\alpha)^2},$$

$$L_{\alpha\alpha} = \frac{2\beta^2}{(\beta-\alpha)^3}, \qquad L_{\beta\beta} = \frac{2\alpha^2}{(\beta-\alpha)^3}.$$

From (3.1) and (3.2), we have the following identities.

(3.3) $(\beta - \alpha) L_\alpha = L$, $\beta(\beta - \alpha)L_\beta = (\beta - 2\alpha)L$,

$(\beta - \alpha)^2 L_{\alpha\alpha} = 2L$, $\beta^2((\beta - \alpha)^2 L_{\beta\beta} = 2\alpha^2 L$.

Now, using (3.2) in (2.3), we get

$$2(2\alpha^3 b^2 + \beta^3 - 3\alpha\beta^2)\left(C + \frac{\alpha r_{00}}{2\beta}\right) = \alpha(\beta - \alpha)\,[\beta r_{00} - 2\alpha(\beta - 2\alpha)s_0].$$

Using this value in (1.8), we have

$$\beta(2\alpha^3 b^2 + \beta^3 - 3\alpha\beta^2)\{\beta(\alpha^2\gamma^i_{00} - \gamma_{000}y^i) + 2\alpha^3(\beta - 2\alpha)s^i_0\}$$
$$+\alpha\{\beta r_{00} - 2\alpha(\beta - 2\alpha)s_0\}(\alpha^2 b^i - \beta y^i) = 0,$$

which can be rearranged as follows

$$(3.4)\quad \{\beta(2\alpha^2 b^2 - 3\beta^2)(\alpha^2\gamma^i_{00} - \gamma_{000}y^i) + 2\alpha^2(6\alpha^2\beta^2 + \beta^4 - 4\alpha^4 b^2)s^i_0$$
$$+ (\beta r_{00} + 4\alpha^2 s_0)(\alpha^2 b^i - \beta y^i)\}\alpha + \{\beta^4(\alpha^2\gamma^i_{00} - \gamma_{000}y^i)$$
$$+ 2\alpha^4\beta(2\alpha^2 b^2 - 5\beta^2)s^i_0 - 2\alpha^2\beta s_0(\alpha^2 b^i - \beta y^i)\} = 0,$$

which is of the type $P^i\alpha + Q^i = 0$ where

$$(3.5)\quad P^i = \beta(2\alpha^2 b^2 - 3\beta^2)(\alpha^2\gamma^i_{00} - \gamma_{000}y^i)$$
$$+ 2\alpha^2(6\alpha^2\beta^2 + \beta^4 - 4\alpha^4 b^2)s^i_0$$
$$+ (\beta r_{00} + 4\alpha^2 s_0)(\alpha^2 b^i - \beta y^i).$$

$$(3.6)\quad Q^i = \beta^4(\alpha^2\gamma^i_{00} - \gamma_{000}y^i) + 2\alpha^4\beta(2\alpha^2 b^2 - 5\beta^2)s^i_0$$
$$- 2\alpha^2\beta s_0(\alpha^2 b^i - \beta y^i).$$

From (3.5) and (3.6) we observe that both P^i and Q^i are rational function in (x^i, y^i) and α is irrational function in (x^i, y^i). Hence $P^i\alpha + Q^i = 0$ will satisfy if and only if $P^i = 0$ and $Q^i = 0$.

From $Q^i = 0$ we observe that $\beta^4\gamma_{000}y^i$ has a factor α^2. Therefore there exist functions $\lambda_i(x)$ such that

$$(3.7)\quad \gamma_{000} = \alpha^2\lambda_0.$$

Using (3.7) in $P^i = 0$, we observe that $\beta r_{00}y^i$ has a factor α^2. Therefore there exist a function $\mu(x)$ such that

$$(3.8)\quad r_{00} = \alpha^2\mu(x).$$

Using (3.7) and (3.8) in $Q^i = 0$, we get

$$(3.9)\quad \beta^3(\gamma^i_{00} - \lambda_0 y^i) + 2\alpha^2(2\alpha^2 b^2 - 5\beta^2)s^i_0 - 2s_0(\alpha^2 b^i - \beta y^i) = 0.$$

From (3.9) we observe that $\beta^3(\gamma^i_{00} - \lambda_0 y^i) + 2\beta s_0 y^i$ has a factor α^2. Therefore there exist a function $\nu^i(x)$ such that

$$(3.10)\quad \beta^3(\gamma^i_{00} - \lambda_0 y^i) + 2\beta s_0 y^i = \alpha^2\nu^i(x).$$

Contracting (3.10) by $y_i = a_{ir}y^r$ and using (3.7), we get $\nu_0 = 2\beta s_0$, which gives $\nu^i = 2(b^i s_0 + \beta s^i)$. Using this in (3.10), we get

$$(3.11)\quad \beta^3(\gamma^i_{00} - \lambda_0 y^i) - 2s_0(\alpha^2 b^i - \beta y^i) = 2\alpha^2\beta s^i.$$

Furthermore using (3.11) in (3.9), we get

(3.12) $$s_0^i = \frac{\beta s^i}{(5\beta^2 - 2\alpha^2 b^2)}.$$

Contracting (3.12) by y_i and using $s_0^i y_i = 0$, we get $\frac{\beta s_0}{(5\beta^2 - 2\alpha^2 b^2)} = 0$, which gives $s_0 = 0$. Hence $s^i = 0$, and therefore (3.11) gives

(3.13) $$\gamma_{00}^i = \lambda_0 y^i.$$

This equation gives

(3.14) $$\gamma_{jk}^i = \lambda_k \delta_j^i + \lambda_j \delta_k^i,$$

which shows that the associated Riemannian space of Finsler space with metric $\frac{\beta^2}{\beta - \alpha}$ is projectively flat.

Since $s_0 = 0$ implies that $s^i = 0$, therefore (3.12)ives $s_0^i = 0$, $s_{ij} = 0$. Using these results and equations (3.7), (3.13) in (3.5) we get $r_{00} = 0$ which gives $r_{ij} = 0$. Hence from (1.1) we get $b_{i;j} = 0$.

Conversely, assume that the associated Riemannian space of Finsler space with metric $\frac{\beta^2}{\beta - \alpha}$ is projectively flat and $b_{i;j} = 0$. Then from (1.1) we get $r_{ij} = 0$, $s_{ij} = 0$ which gives $r_{00} = 0$, $s_0^i = 0$, $s_0 = 0$. Using all these results with equation (3.13), we see that equation (3.4) is satisfied identically. Therefore the Finsler space is projectively flat. Summarizing all these results we have the following.

Theorem (3.1). A Finsler space with (α, β) metric $L(\alpha, \beta) = \frac{\beta^2}{\beta - \alpha}$ is projectively flat if and only if the associated Riemannian space is projectively flat and $b_{i;j} = 0$.

V. Projectively flat Finsler space with metric $L(\alpha, \beta) = \alpha + \frac{\beta^2}{\alpha}$.

Consider a Finsler space with metric

(4.1) $$L(\alpha, \beta) = \alpha + \frac{\beta^2}{\alpha}.$$

Then we have the following:

(4.2) $$L_\alpha = \frac{\alpha^2 - \beta^2}{\alpha^2}, \qquad L_\beta = \frac{2\beta}{\alpha}, \qquad L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}, \qquad L_{\beta\beta} = \frac{2}{\alpha}.$$

Using (4.2) in equation (2.3), we get

(4.3) $$\{\alpha^2(1 + b^2) - 3\beta^2\}\left(C + \frac{\alpha r_{00}}{2\beta}\right) = \frac{\alpha}{2\beta}\{(\alpha^2 - \beta^2) r_{00} - 4\alpha^2 s_0\}.$$

Using (4.2) and (4.3) in equation (1.8), we get

(4.4) $$\{\alpha^2(1 + b^2) - 3\beta^2\}\{\alpha^2 - \beta^2)(\alpha^2 \gamma_{00}^i - \gamma_{000} y^i) + 4\alpha^4 \beta s_0^i\}$$

$$+\ 2\alpha^2\beta\{(\alpha^2-\beta^2)r_{00}-4\alpha^2 s_0\}(\alpha^2 b^i-\beta y^i) = 0.$$

From (4.4) we observe that $\beta^4\gamma_{000}\, y^i$ has a factor α^2. Therefore there exist a function $\lambda_i(x)$ such that

(4.5) $\gamma_{000} = \alpha^2 \lambda_0$.

Using (4.5) in (4.4) it reduces to

(4.6) $$\{\alpha^2(1+b^2)-3\beta^2\}\{\alpha^2-\beta^2)(\gamma^i_{00}-\lambda_0 y^i)+4\alpha^4\beta s^i_0\}$$
$$+2\beta\{(\alpha^2-\beta^2)r_{00}-4\alpha^2 s_0\}(\alpha^2 b^i-\beta y^i) = 0.$$

From (4.6) we again observe that

$$\{\alpha^2(1+b^2)-3\beta^2\}s^i_0-2s_0(\alpha^2 b^i-\beta y^i)$$

has a factor $(\alpha^2-\beta^2)$. Therefore there exist a function $\mu^i(x)$ such that

(4.7) $$\{\alpha^2(1+b^2)-3\beta^2\}s^i_0-2s_0(\alpha^2 b^i-\beta y^i) = (\alpha^2-\beta^2)\mu^i.$$

Contracting (4.7) by y_i and using $s^i_0 y_i = 0$, $(\alpha^2 b^i-\beta y^i)y_i = 0$, we get $\mu_0 = 0$ which gives $\mu^i = 0$. Using this in (4.7), we get

(4.8) $$\{\alpha^2(1+b^2)-3\beta^2\}s^i_0 = 2s_0(\alpha^2 b^i-\beta y^i).$$

Contracting (4.8) by b_i and using $s^i_0 b_i = s_0$, we get

$$\{\alpha^2(1-b^2)-\beta^2\}s_0 = 0,$$

which gives $s_0 = 0$. Hence from (4.8), we get $s^i_0 = 0$, $s_{ij} = 0$, which implies that b_i is a gradient of some scalar function b(x).

Furthermore using $s^i_0 = 0$ and $s_0 = 0$ in (4.6) we get

(4.9) $$\{\alpha^2(1+b^2)-3\beta^2\}(\gamma^i_{00}-\lambda_0 y^i)+2\beta r_{00}(\alpha^2 b^i-\beta y^i) = 0.$$

Contracting (4.9) by b_i we get

(4.10) $$\{\alpha^2(1+b^2)-3\beta^2\}(\gamma^i_{00}b_i-\lambda_0\beta)+2\beta r_{00}(\alpha^2 b^2-\beta^2) = 0.$$

This equation determines the Christofell symbol of associated Riemannian space. Summarizing above all we have following

Theorem (4.1). If a Finsler space with (α, β) metric $\alpha + \dfrac{\beta^2}{\alpha}$ is projectively flat, then equation 4.10) is satisfied and b_i is a gradient of some scalar function b(x).

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