

Generalized Simpson-Newton's Method for Solving Nonlinear Equations with Cubic Convergence

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Abstract: In the recent past, different variants of Newton's method with cubic convergence have become popular iterative methods to find the roots of non-linear equations. In this paper, a new class of Newton's method for solving a single nonlinear equation is proposed. This method is the generalization of Simpson's integration rule applied on the Newton's theorem. Some of the existing methods become special cases of this method. Third order convergence of the proposed methods is established. Numerical examples are provided. A comparison study is done to show the efficiency of this method for different parameters in the method.

Keywords: Iterative Method, Newton's Method, Non-linear equation, Third order convergence, Simpson integration method.

I. Introduction

A frequently occurring and most important problem of applied mathematics and engineering is to find a root of the equation

$$f(x) = 0 \quad (1)$$

where $f: D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ for an open interval D is a scalar function. Newton's method is one of the famous iterative methods to solve equation (1). It is well known that it has quadratic convergence. The iterative formula of Newton's method is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \quad (2)$$

By improving Newton's method, Chebyshev [1] derived a cubic convergent iterative method which requires computing of second derivative of the function (1). Weerakoon et al [2] improved the order of convergence to three via trapezoid rule on Newton's theorem and obtained Arithmetic Newton's method. Hasanov et al [3] modified Newton's method with a third order convergent method by using Simpson's rule. Nedzhibov [4] gave many classes of iterative methods using different quadrature rules. Later many authors [5, 6, 7, 9] proposed different variants of Newton's method of order three. Babajee et al [8] took eight different variants of Newton's method and analyzed the relationship between them. They explored the properties of arithmetic mean Newton's method and explained third order accuracy via the Taylor's series expansion. Tibor et al [10] suggested geometric mean Newton's method for simple and multiple roots and showed this method is of order three for simple roots and order one for multiple roots. Ababneh [11] modified Newton's method using contra-harmonic mean instead of Arithmetic mean. The advantage of all the above methods is that they require only the first derivative of the function and has cubic convergence.

In this paper, we propose a generalization of the Simpson-Newton's method [3, 12] where Simpson's integration rule is applied for approximating the integral in the Newton's theorem. We find that some of the existing methods are special cases of the proposed method. Third order convergence of the proposed method is established. In Section II, we present some definitions related to our study. Some known variants of Newton's method are discussed in Section III. Section IV presents the new method and its analysis of convergence and Section V gives numerical results. Finally, a discussion is carried out on the merits and demerits of the proposed method in Section VI.

II. Preliminaries

Definition 2.1 [2]: Let $\alpha \in \mathbb{R}$, $x_n \in \mathbb{R}$, $n = 0, 1, 2, \dots$. Then the sequence $\{x_n\}$ is said to converge to α if $\lim_{n \rightarrow \infty} |x_n - \alpha| = 0$. If, in addition, there exist a constant $c \geq 0$, an integer $n_0 \geq 0$ and $p \geq 0$ such that for all $n > 0$, $|x_{n+1} - \alpha| \leq c|x_n - \alpha|^p$, then $\{x_n\}$ is said to converge to α with order p . If $p = 2$ or 3 , the convergence is said to be quadratic or cubic respectively.

Definition 2.2[2]: Let α be a root of the function (1) and suppose that x_{n-1} , x_n and x_{n+1} are three successive iterations closer to the root α . Then, the computational order of convergence (COC) denoted by ρ can be approximated using the formula $\rho \approx \frac{\ln |(x_{n+1} - \alpha)/(x_n - \alpha)|}{\ln |(x_n - \alpha)/(x_{n-1} - \alpha)|}$.

III. Some Known Variants of Newton's Method

Let α be a simple root of a sufficiently differentiable function $f(x)$. Consider the numerical solution of the function $f(x) = 0$. From Newton's theorem, we have

$$f(x) = f(x_n) + \int_{x_n}^x f'(t)dt. \tag{3}$$

Arithmetic Mean Newton's method (AN): By using the trapezoidal rule in (3), we obtain

$$\int_{x_n}^x f'(t)dt \approx \frac{(x-x_n)}{2k} \left[f'(x_n) + 2 \sum_{i=1}^{k-1} f' \left(x_n - \frac{if(x_n)}{kf'(x_n)} \right) + f'(x) \right]. \tag{4}$$

From equations (3) and (4), a new approximation x_{n+1} for x when $k = 1$, we get arithmetic mean Newton's method [2]

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)+f'(x_{n+1})}. \tag{5}$$

Since equation (5) is implicit, we can overcome this implicit nature by the use of Newton's iterative step (2). However, we denote x_{n+1} as y_n here-in-afterwards, where

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \dots \tag{6}$$

Midpoint Newton's method (MN): If the integral in (3) is approximated using the midpoint integration rule instead of trapezoidal rule, we get the midpoint Newton's method [6]

$$x_{n+1} = x_n - \frac{f(x_n)}{f' \left(\frac{x_n+y_n}{2} \right)}, \quad n = 0, 1, 2, \dots, \text{ where } y_n \text{ is calculated from (6).}$$

Trapezoidal-Newton's method (TN): From equations (3) and (4), one obtains a new approximation for $k = 2$

$$x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n)+2f' \left(\frac{x_n+y_n}{2} \right) + f'(y_n)},$$

where y_n is calculated from (6). The above method is called as Trapezoidal-Newton's method [4].

Simpson-Newton's method (SN): Simpson's $\frac{1}{3}$ integration rule is used to find the value of the integral in (3) to obtain the Simpson-Newton's method [3] as follows:

$$\int_{x_n}^x f'(t)dt \approx \frac{(x-x_n)}{6} \left[f'(x_n) + 4f' \left(\frac{x+x_n}{2} \right) + f'(x) \right]. \tag{7}$$

Simplifying and computing for x_{n+1} from equation (7), one obtains a new approximation

$$x_{n+1} = x_n - \frac{6f(x_n)}{f'(x_n)+4f' \left(\frac{x_n+x_{n+1}}{2} \right) + f'(x_{n+1})}. \tag{8}$$

Since equation (8) is implicit, replacing x_{n+1} with y_n (calculated from (6)) on the right hand side, we get

$$x_{n+1} = x_n - \left(\frac{6f(x_n)}{f'(x_n)+4f' \left(\frac{x_n+y_n}{2} \right) + f'(y_n)} \right), \quad n = 0, 1, 2, \dots \tag{9}$$

Harmonic-Simpson-Newton's method (HSN): After dividing numerator and denominator in equation (9) by 2 and using harmonic mean instead of arithmetic mean, we obtain Harmonic-Simpson-Newton's method [12]

$$x_{n+1} = x_n - \left(\frac{3f(x_n)}{\frac{2f'(x_n)f'(y_n)}{f'(x_n)+f'(y_n)} + 2f' \left(\frac{x_n+y_n}{2} \right)} \right), \quad n = 0, 1, 2, \dots$$

IV. Generalized Simpson-Newton's Method

Introducing arbitrary parameters m, m_1, m_2 and m_3 in the second term of equation (9), we obtain

$$x_{n+1} = x_n - \left(\frac{mf(x_n)}{m_1f'(x_n)+m_2f' \left(\frac{x_n+y_n}{2} \right) + m_3f'(y_n)} \right), \quad n = 0, 1, 2, \dots, \tag{10}$$

where $m_1+m_2+m_3 = m$ and y_n is calculated from (6).

Remark 1. (a) If $m = m_1=1$ and $m_2 = m_3=0$, we get Newton's method. (b) If $m =2, m_1=m_3=1$ and $m_2 = 0$, we get Arithmetic Newton's method. (c) If $m= m_2= 1$ and $m_1= m_3=0$, we get Midpoint Newton's method. (d) If $m=4, m_1= m_3=1$ and $m_2 = 2$, we get Trapezoidal Newton's method. (e) If $m=6, m_1=m_3= 1$, and $m_2 = 4$, we get Simpson-Newton's method. Hence, we call this method (10) **Generalized Simpson-Newton's method (GSN)**.

Analysis of Convergence

In the following theorem, we establish the cubic convergence of the present method (GSN).

Theorem 4.1: Let $\alpha \in D$, be a simple zero of a sufficiently differentiable function $f: D \subseteq R \rightarrow R$ for an open interval D . If x_0 is sufficiently closer to α , then the method (10) has third order convergence.

Proof. Let α be a simple zero of function $f(x) = 0$. (That is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$).

By expanding $f(x_n)$ and $f'(x_n)$ by Taylor series about α , we obtain

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4) \dots], \tag{11}$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4) \dots], \tag{12}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$, $k = 2, 3, \dots$ and $e_n = x_n - \alpha$. Dividing equation (11) by (12), we get

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 + 2(c_2^2 - c_3) e_n^3 + O(e_n^4). \text{ Hence, we get from (6)}$$

$$y_n = \alpha + c_2 e_n^2 + 2(c_3 - c_2^2) e_n^3 + O(e_n^4). \tag{13}$$

Also, we have

$$\frac{y_n + x_n}{2} = \alpha + \frac{1}{2} e_n + \frac{1}{2} c_2 e_n^2 + (c_3 - c_2^2) e_n^3 + O(e_n^4). \tag{14}$$

Expanding $f'(y_n)$ by Taylor's series about α and using (13), we get

$$f'(y_n) = f'(\alpha) [1 + 2c_2^2 e_n^2 + O(e_n^3)]. \tag{15}$$

Again expanding $f'(\frac{x_n + y_n}{2})$ by Taylor's series about α and using (14), we get

$$f'(\frac{x_n + y_n}{2}) = f'(\alpha) [1 + c_2 e_n + (c_2^2 + \frac{3}{4} c_3) e_n^2 + O(e_n^3)]. \tag{16}$$

From (12), (15), (16) and taking $m_1 + m_2 + m_3 = m$, we have

$$m_1 f'(x_n) + m_2 f'(\frac{x_n + y_n}{2}) + m_3 f'(y_n) = m f'(\alpha) [1 + c_2 (\frac{2m_1 + m_2}{m}) e_n + (c_2^2 (\frac{2m_1 + m_2}{m}) + 3c_3 (\frac{m_1 + m_2}{m})) e_n^2 + O(e_n^3)]. \tag{17}$$

Then from equation (11) and (17), we get

$$\frac{m f(x_n)}{m_1 f'(x_n) + m_2 f'(\frac{x_n + y_n}{2}) + m_3 f'(y_n)} = e_n + e_n^2 [c_2 - \frac{c_2}{m} (2m_1 + m_2)] + e_n^3 [c_3 (1 - 3 (\frac{m_1 + m_2}{m})) + c_2^2 (-2 + \frac{2m_1 + m_2}{m})] + O(e_n^4). \tag{18}$$

The above equation (18) is substituted in (10) to get

$$x_{n+1} = x_n - [e_n + e_n^2 [c_2 - \frac{c_2}{m} (2m_1 + m_2)] + e_n^3 [c_3 (1 - 3 (\frac{m_1 + m_2}{m})) + c_2^2 (-2 + (\frac{2m_1 + m_2}{m})^2)]] + O(e_n^4). \tag{19}$$

Since $x_n = e_n + \alpha$, we finally obtain from (19)

$$e_{n+1} = e_n^2 [-c_2 + \frac{c_2}{m} (2m_1 + m_2)] + e_n^3 [c_3 (-1 + 3 (\frac{m_1 + m_2}{m})) + c_2^2 (2 - (\frac{2m_1 + m_2}{m})^2)] + O(e_n^4). \quad \square$$

Remark 2. It can be easily verified that the coefficient of e_n^2 will be zero in the above equation for all the variants of Newton's method given in Section III. Hence, cubic convergence is established for GSN for all those parameters (m_1 , m_2 and m) satisfying the condition $2m_1 + m_2 = m$.

V. Numerical Examples

Table 1

$f(x)$	X_0	Number of iterations						Root
		NM	AN	MN	TN	SN	HSN	
$f_1(x) = x^3 + x^2 - 2$	1.5	6	5	4	5	5	4	$\alpha = 1$
	3	8	6	6	6	6	5	
$f_2(x) = \sin x + x \cos x$	1.3	6	5	5	5	5	5	$\alpha = 2.028757838110434$
	3	6	5	4	4	4	4	
$f_3(x) = x^2 - e^x - 3x + 2$	3	7	5	5	5	5	5	$\alpha = 0.257530285439861$
	5	9	7	6	6	6	6	
$f_4(x) = x e^x - 1$	3	9	7	6	7	6	6	$\alpha = 0.567143290409784$
	4.5	11	8	8	8	8	7	
$f_5(x) = x^3 - 10$	3	6	5	5	5	5	4	$\alpha = 2.154434690031884$
	-1	7	NC	6	11	7	5	
$f_6(x) = e^x + x - 20$	10	13	9	9	9	9	8	$\alpha = 2.842438953784447$
	5	8	6	4	6	6	5	

In this section, we present the numerical results of the proposed method (GSN). Also, we examine GSN for different values of m , m_1 , m_2 and m_3 . Different functions and number of iterations to find their roots are given in **Table 1**. Computational Order of Convergence (COC) and the Number of Function Evaluation (NOFE) are given in **Table 2**. We note that the function $f_5(x)$ is not convergent for $x_0 = -1$, by AN even with 100 iterations but convergent for all other cases. Numerical computations are done in the Matlab software with

double precision accuracy and the results are presented. We have used the following **stopping criteria** for the iterative process:

$$|x_{n+1} - x_n| + |f(x_{n+1})| < \varepsilon, \text{ where } \varepsilon = 10^{-14}.$$

Table 2

$f(x)$	X_0	COC						NOFE					
		NM	AN	MN	TN	SN	HSN	NM	AN	MN	TN	SN	HSN
$f_1(x)$	1.5	2.00	2.99	2.75	2.99	2.99	2.80	12	15	12	20	20	16
	3	2.00	2.98	2.99	2.99	2.99	2.80	16	18	18	24	24	20
$f_2(x)$	1.3	1.99	3.82	3.17	3.07	3.12	3.08	12	15	15	20	20	20
	3	1.99	4.10	2.71	2.88	2.60	2.70	12	15	12	16	16	16
$f_3(x)$	3	2.00	3.04	3.39	2.91	2.66	2.78	14	15	15	20	20	20
	5	2.00	3.01	3.29	3.20	3.16	2.69	18	21	18	24	24	24
$f_4(x)$	3	2.00	2.99	2.88	2.99	2.84	2.93	18	21	18	28	24	24
	4.5	2.00	2.96	2.99	2.98	2.99	2.91	22	24	24	32	32	28
$f_5(x)$	3	2.00	2.99	2.99	2.99	2.99	2.81	12	15	15	20	20	16
	-1	2.00	NC	2.91	3.01	3.02	2.95	14	NC	18	44	36	20
$f_6(x)$	10	2.00	2.93	3.00	2.98	2.99	2.94	26	27	27	36	28	32
	5	2.00	2.99	3.00	2.99	2.99	2.94	16	18	12	24	24	20

NC – Not Convergent

VI. Discussion

The basic advantage of the present method is that it is a generalized class of Newton's method using the Simpson's formula applied on the Newton's theorem. The proposed new class of methods has the advantage of evaluating only the first derivative of $f(x)$, third order convergence and requires less number of iterations to achieve the desired accuracy. Also, we find that (see Remark 1) some of the existing methods given in Section III are special cases of the proposed GSN. Few methods like Trapezoidal and Simpson Newton's methods require four function evaluation which is one extra compared to other third-order convergent Newton methods. However, this drawback in efficiency is compensated by the decrease in the number of iterations. The motivation behind this contribution is to bring in certain existing methods under one class of method, the Generalized Simpson Newton's method.

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