On Generalized $\psi-|C, \alpha, \beta, \gamma, \delta|_k$-Summability Factor

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Abstract: In this paper we have established a theorem on $\psi-|C, \alpha, \beta, \gamma, \delta|_k$-summability factor, which gives some new results.

I. Introduction

A positive sequence $(b_n)$ is said to be almost increasing if there exist a positive sequence $(c_n)$ and two positive constants $A$ and $B$ such that $Ac_n \leq b_n \leq Bc_n$ (Bari [2]).

A sequence $(\lambda_n)$ is said to be of bounded variation, denoted by $(\lambda_n) \in BV$ if

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty.$$  

A positive sequence $X = (X_n)$ is said to be quasi-$\sigma$-power increasing sequence if there exist a constant $k = k(\sigma, X) \geq 1$ such that $kn^\sigma X_n \geq m^\sigma X_m$, $n \geq m \geq 1$ (Leindler [7]).

Let $\psi_n$ be a sequence of complex numbers. Let $\sum a_n$ be a given infinite series with partial sum $(s_n)$.

We denote by $z_{\alpha, \beta}^n$ and $t_{\alpha, \beta}^n$ the $n^{th}$ Cesaro means of order $(\alpha, \beta)$ with $\alpha + \beta > -1$ of the sequences $(s_n)$ and $(na_n)$ respectively (Borwein [5]).

$$z_{\alpha, \beta}^n = \frac{1}{A_{\alpha, \beta}} \sum_{v=0}^{n} A_{\alpha, \beta}^v a_v s_v$$

$$t_{\alpha, \beta}^n = \frac{1}{A_{\alpha, \beta}} \sum_{v=0}^{n} A_{\alpha, \beta}^v a_v v a_v$$

where $A_{\alpha, \beta}^n = O(n^{\alpha+\beta})$, $\alpha + \beta > -1$, $A_{\alpha}^0 = 1$, $A_{\alpha}^\infty = 0$ for $n > 0$.

The series $\sum \psi_n$ is said to be summable $|C, \alpha, \beta|_k$, $k \geq 1$ and $\alpha + \beta > -1$ (Das [6]) if

$$\sum_{n=1}^{\infty} |\psi_n z_{\alpha, \beta}^n|^{\gamma} < \infty$$

The series $\sum \psi_n$ is said to be summable $|C, \alpha, \beta, \gamma, \delta|_k$, $k \geq 1$, $\alpha + \beta > -1$, $\delta \geq 0$ and $\gamma$ is a real number (Bor [4]) if

$$\sum_{n=1}^{\infty} \frac{1}{n^{\gamma(\delta+k-1)}} |\psi_n z_{\alpha, \beta}^n|^{\gamma} < \infty$$

The series $\sum \psi_n$ is said to be summable $\psi-|C, \alpha|_k$, $k \geq 1$, $\alpha > -1$ if (Balci [1])

$$\sum_{n=1}^{\infty} |\psi_n z_{\alpha, \beta}^n - z_{\alpha, \beta}^{n-1}|^k = \sum_{n=1}^{\infty} n^{-k} |\psi_n \alpha_n|^k < \infty$$

And the series $\sum \psi_n$ is said to be summable $\psi-|C, \alpha, \beta, \gamma, \delta|_k$ if

$$\sum_{n=1}^{\infty} \frac{1}{n^{\gamma(\delta+k-1)-1}} |\psi_n t_{\alpha, \beta}^n|^k < \infty$$

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II. Known theorem

Tuncer has proved the following theorem

**Theorem 2.1** Let \( k \geq 1, 0 \leq \delta < \alpha \leq 1 \) and \( \gamma \) be a real number such that \((\alpha + \beta + 1 - \gamma(\delta + 1))k > 1 \) and let the sequences \( (B_n) \) and \( (\lambda_n) \) such that \((B_n)\) is \( \delta \)-quasi-monotone with

\[
|\Delta \lambda_n| \leq B_n, \quad \lambda_n \to 0 \text{ as } n \to \infty
\]

(1)

\[
\sum_{n=1}^{\infty} n\delta_n \log n < \infty \quad \text{and} \quad nB_n \log n
\]

(2)

is convergent. If the sequence \( (W_n^{\alpha,\beta}) \) defined by

\[
W_n^{\alpha,\beta} = \max_{1 \leq i \leq n} |t_i^{\alpha,\beta}|, \quad 0 < \alpha < 1, \beta > -1
\]

(3)

satisfies the condition

\[
\sum_{n=1}^{m} \frac{n^{\gamma(k+1)}}{n^k} (W_n^{\alpha,\beta})^k = O(\log m) as \to \infty
\]

(4)

are satisfied then the series \( \sum a_n \lambda_n \) is summable \( |C, \alpha, \beta, \gamma, \delta|_k \).

III. The main result

The aim of this paper is to generalize Theorem 2.1 to \( \psi - |C, \alpha, \beta, \gamma, \delta|_k \) summability. We shall prove the following theorem.

**Theorem 3.1** Let \( \psi_n \) be the sequence of Complex numbers and let the sequence \( (B_n) \) & \( (\lambda_n) \) such that the conditions (1), (2), (3), (4) with

\[
\sum_{n=1}^{m} \frac{n^{\gamma(2k+1)}}{n^k} (\psi_n^{\alpha,\beta})^k = O(\log m) as \to \infty
\]

are satisfied then the series \( \sum a_n \lambda_n \) is summable \( \psi_n - |C, \alpha, \beta, \gamma, \delta|_k \).

IV. Lemmas

We need the following lemmas for the proof of our theorem

**Lemma 4.1** (Mazhar [9]) Under the condition on \( (B_n) \) as taken in the statement of the theorem, we have following

\[
nB_n \log n = O(1)
\]

(6)

and

\[
\sum_{n=1}^{m} n \log n |\Delta B_n| < \infty
\]

(7)

**Lemma 4.2** (Bor [4]) If \( 0 < \alpha \leq 1, \beta > -1 \) and \( 1 \leq v \leq n \) then

\[
\left| \sum_{p=v}^{n} A_{n-p}^{\alpha-1} A_p^\beta a_p \right| \leq \max_{1 \leq n \leq v} \sum_{p=0}^{m} A_{n-p}^{\alpha-1} A_p^\beta a_p
\]

(8)

V. Proof of the theorem

Let \( (T_n^{\alpha,\beta}) \) be the \( n \)th \( (C, \alpha, \beta) \) mean of the sequence \( (na_n \lambda_n) \) then we have

\[
T_n^{\alpha,\beta} = \frac{1}{A_{n+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta v a_v \lambda_v.
\]

Using Abel's transformation.
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$$T^{\alpha, \beta}_n = \frac{1}{A^{\alpha + \beta}_n} \sum_{v=1}^{n-1} A_v^{\alpha, \beta} \sum_{p=1}^{v} A^{\alpha - 1}_{n-p} A^{\beta}_p \cdot \lambda_p$$

$$+ \frac{\lambda_n}{A^{\alpha + \beta}_n} \sum_{v=1}^{n} A^{\alpha - 1}_{n-v} A^{\beta}_v \cdot \alpha_v$$

$$|T^{\alpha, \beta}_n| \leq \frac{1}{A^{\alpha + \beta}_n} \sum_{v=1}^{n-1} |A_v^{\alpha, \beta}| \left| \sum_{p=1}^{v} A^{\alpha - 1}_{n-p} A^{\beta}_p \cdot \lambda_p \right|$$

$$+ \frac{|\lambda_n|}{A^{\alpha + \beta}_n} \sum_{v=1}^{n} A^{\alpha - 1}_{n-v} A^{\beta}_v \cdot \alpha_v$$

$$\leq \frac{1}{A^{\alpha + \beta}_n} \sum_{v=1}^{n-1} A^{\alpha, \beta}_v W^{\alpha, \beta}_v |A_v^{\alpha, \beta}| + |\lambda_n| W^{\alpha, \beta}_n$$

$$= T^{\alpha, \beta}_{n,1} + T^{\alpha, \beta}_{n,2} \ (\text{say})$$

Since

$$|T^{\alpha, \beta}_{n,1} + T^{\alpha, \beta}_{n,2}| \leq 2^k \left( |T^{\alpha, \beta}_{n,1}| + |T^{\alpha, \beta}_{n,2}| \right)$$

In order to complete the proof of theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{(\phi + k-1)-k} |T^{\alpha, \beta}_n \cdot \psi^n| < \infty, r = 1, 2$$

Whenever $k > 1$, we can apply Hölder's inequality with indices $k$ and $k'$, where $\frac{1}{k} + \frac{1}{k'} = 1$ we get that

$$\sum_{n=1}^{\infty} n^{(\phi + k-1)-k} |T^{\alpha, \beta}_n \cdot \psi^n|$$

$$\leq \sum_{n=1}^{\infty} n^{(\phi + k-1)-k} \left| \sum_{v=1}^{\infty} A^{\alpha, \beta}_v W^{\alpha, \beta}_v \Delta A_v \right|$$

$$= O(1) \sum_{n=1}^{\infty} \left| \sum_{v=1}^{\infty} A^{\alpha, \beta}_v W^{\alpha, \beta}_v \right| \int_0^{\infty} \frac{dx}{x^{(\phi + k-1)-k}}$$

$$= O(1) \sum_{v=1}^{\infty} \left| A^{\alpha, \beta}_v \right| \left| W^{\alpha, \beta}_v \right| \int_0^{\infty} \frac{dx}{x^{(\phi + k-1)-k}}$$

$$= O(1) \sum_{v=1}^{\infty} \left| A^{\alpha, \beta}_v \right| \left| B_v \right| \left( W^{\alpha, \beta}_v \right)^k |\psi_v|^k$$

$$= O(1) \sum_{v=1}^{\infty} \left| A^{\alpha, \beta}_v \right| \left| B_v \right| \left( W^{\alpha, \beta}_v \right)^k |\psi_v|^k$$

$$+ O(1) \sum_{v=1}^{\infty} \left| A^{\alpha, \beta}_v \right| \left| B_v \right| \left( W^{\alpha, \beta}_v \right)^k |\psi_v|^k$$

$$= O(1) \sum_{v=1}^{\infty} \left| A^{\alpha, \beta}_v \right| \left| B_v \right| \log v + O(1) m |B_m| \log m$$
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$$= O(1) \sum_{v=1}^{m-1} \Delta B_v |\log v + O(1) \sum_{v=1}^{m-1} |B_{v+1} |\log v + O(1)m |B_m |\log m$$

$$= O(1) \text{ as } m \to \infty$$

$$\sum_{n=2}^{m+1} n^{(\beta k - 1) - k} \left| T_{n,2}^{\alpha, \beta} \psi_n \right|^k = O(1) \sum_{n=1}^{m} \lambda_n n^{(\beta k - 1) - k} (W_n^{\alpha, \beta})^k |\psi_n|^k$$

$$= O(1) \sum_{n=1}^{m} \Delta \lambda_n \sum_{v=1}^{n} n^{(\beta k - 1) - k} (W_v^{\alpha, \beta})^k |\psi_n|^k$$

$$+ O(1) \lambda_m \sum_{v=1}^{m} n^{(\beta k - 1) - k} (W_v^{\alpha, \beta})^k |\psi_n|^k$$

$$= O(1) \sum_{n=1}^{m} \Delta \lambda_n |\log n + O(1) |\lambda_m |\log m$$

$$= O(1) \sum_{n=1}^{m} |B_n |\log n + O(1) |\lambda_m |\log m$$

$$= O(1) \text{ as } m \to \infty$$

by virtue of the hypothesis of the theorem and lemma 1. This completes the proof of the theorem.

References