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On Boundedness and Compositions of the Operator $(G_{\rho,\eta,\gamma,\omega;a+\Psi})(x)$ and the Inversion Formula

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Abstract: *In this paper we established Boundedness and Compositions of the Operator $(G_{\rho,\eta,\gamma,\omega;a+\Psi})(x)$ and the Inversion Formulae on the space L(a,b) and C[a,b] , given by Hartely and Lorenzo[5]*
Key words *-$G_{\rho,\eta,r}[a,z]$ function, Riemann liouville fractional integral, Riemann liouville fractional derivative , beta-integral.*

I. Introduction and preliminaries

The Riemann-Liouville fractional integral $I_{a^+}^{\alpha}$ and fractional derivative $D_{a^+}^{\alpha}$ [2], [4] of order $\alpha > 0$ are given by $\left(I_{a^+}^{\alpha}F\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{F(t)}{(x-t)1-\alpha}dt$ 1.1

where $\alpha \epsilon$ C , Re$(\alpha) > 0$ and

$\left(D_{a^+}^{\alpha}F\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a^+}^{\alpha-n}F\right)(x)$ 1.2

where $\alpha \epsilon$ C , Re(α) > 0 (n=$|\mathrm{Re}(\alpha)| + 1$).

G function introduced by Lorenzo and Hartley[1],[5]

$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(r)_n (az^{\rho})^n}{\Gamma(\rho n+\rho r-\eta)n!}$ 1.3

where $(r)_n$ is the Pochhammer symbol and Re$(\rho r - \eta)$

where q, γ, δ ϵ C, Re $(q > 0)$ and Re $(\gamma > 0)$, Re $(q\gamma - \delta) > 0$

Properties of function $G_{\rho,\eta,\gamma}[a,z]$

For $\rho,\ \eta,\ \omega,\ \gamma,\ \sigma,\ q,\ \alpha \in C,\ (\mathrm{Re}\ (\rho), \mathrm{Re}\ (\eta), \mathrm{Re}(q), \mathrm{Re}\ (\alpha) > 0)$ and $n \in N$ there hold the following properties for the special function $G_{\rho,\eta,\gamma}[a,z]$ defined in (1.3) are given by H.Nagar et al.[3]

Property-1 $\left(\frac{d}{dz}\right)^n \left(G_{\rho,\eta,\gamma}[\omega,\ z]\right) = G_{\rho,\eta+n,\gamma}[\omega,z]$ 1.4

Property-2 $\int_0^x G_{\rho,\eta,\gamma}[\omega,(x-t)]\,G_{\rho,q,\sigma}[\omega,t]\,dt = G_{\rho,\eta+q,\gamma+\sigma}[\omega,x]$ 1.5

Property-3 $\left(I_{a+}^{\alpha}\,G_{\rho,\eta,\gamma}[\omega,(t-a)]\right)(x) = G_{\rho,\eta-\alpha,\gamma}[\omega,(x-a)],\quad (x>a)$ 1.6

Property-4 $D_{a+}^{\alpha}\ \left(G_{\rho,\eta,\gamma}[\omega,(t-a)]\right)(x) = G_{\rho,\eta+\alpha,\gamma}[\omega,(x-a)],\quad (x>a)$ 1.7

The fractional Integral operators $G_{\rho,\eta,\gamma,\omega;a+}$

$$\left(G_{\rho,\eta,\gamma,\omega\ ;a+}\psi\right)(x) = \int_a^x G_{\rho,\eta,\gamma}[\omega,\ (x-t)]\,\psi(t)dt,\ (x>a) \qquad 1.8$$

where $\rho,\ \eta,\ \gamma,\ \omega\ \in C,\ (\mathrm{Re}(\rho),\ \mathrm{Re}(\eta) > 0)$

II. Boundedness of Operator $(G_{\rho,\eta,\gamma,\omega;a+\Psi})(x)$

In the following theorems we prove the boundedness on the space L(a,b) and C[a,b] of the operator $(G_{\rho,\eta,\gamma,\omega;a+\Psi})(x)$ defined in (1.8).

Theorem-1 Let $\rho, \gamma, \eta, \omega \in C, \left(\mathrm{Re}(\rho), \mathrm{Re}(\eta) > 0\right)$ and $b > a$ then the operator $G_{\rho,\eta,\gamma,\omega;a+}$ is bounded on $L(a, b)$ and $\left\| G_{\rho,\eta,\gamma,\omega;\, a+} \psi \right\|_1 \leqq B^* \, \| \psi \|_1$ 2.1

Where $B^* = (b-a)^{\mathrm{Re}(\rho)\gamma - \mathrm{Re}(\eta)} \sum\limits_{k=0}^{\infty} \dfrac{|(\gamma)_k| \left|\omega(b-a)^{\mathrm{Re}(\rho)}\right|^k}{|\Gamma(\rho\gamma + \rho k - \eta)| \, [\mathrm{Re}(\rho)(\gamma + k) - \mathrm{Re}(\eta)] \, k!}$ 2.2

Theorem-2 Let $\rho, \gamma, \eta, \omega \in C, \left[\mathrm{Re}(\rho), \mathrm{Re}(\eta) > 0\right]$ and $b > a$ then the operator $G_{\rho,\eta,\gamma,\omega\,;\,a+}$ is bounded on $C[a, b]$ and $\left\| G_{\rho,\eta,\gamma,\omega\,;a+} \psi \right\|_C \leqq B^* \, \| \psi \|_C$

where B^* is given in (2.2)

Proof of Theorems-1, 2

To prove Theorem-1, we denote its left hand side by Δ_4 i.e. $\Delta_4 = \left\| G_{\rho,\eta,\gamma,\omega;a+} \psi \right\|_1$. Now using the definition of operator $G_{\rho,\eta,\gamma,\omega;a+}$ in (1.8) and definition of the function $G_{\rho,\eta,\gamma}[\omega, (x-t)]$ in (1.3) and then on interchanging the order of integration by the Dirichlet formula we have,

$$\Delta_4 = \int_a^b \left| \int_a^x (x-t)^{\rho\gamma - \eta - 1} \sum_{k=0}^{\infty} \frac{(\gamma)_k \left[\omega(x-t)^{\rho}\right]^k}{k!\, \Gamma(\rho\gamma + \rho k - \eta)} \psi(t)\, dt \right| dx$$

$$\leqq \int_a^b \left| \int_t^b (x-t)^{\mathrm{Re}(\rho)\gamma - \mathrm{Re}(\eta) - 1} \sum_{k=0}^{\infty} \frac{(\gamma)_k\, \omega^k\, (x-t)^{\mathrm{Re}(\rho)k}}{k!\, \Gamma(\rho\gamma + \rho k - \eta)} \psi(t)\, dt \right| dx, \quad t > a,\ x \le b;$$

which on putting $x - t = u$ takes the following form :

$$\Delta_4 \leqq \int_a^b \left| \int_0^{b-t} \sum_{k=0}^{\infty} u^{\mathrm{Re}(\rho)(\gamma+k) - \mathrm{Re}(\eta) - 1} \frac{(\gamma)_k\, |\omega|^k}{k!\, |\Gamma(\rho\gamma + \rho k - \eta)|} du \right| |\psi(t)|\, dt$$

$$\leqq \sum_{k=0}^{\infty} \frac{|(\gamma)_k|\, |\omega|^k}{k!\, |\Gamma(\rho\gamma + \rho\, k - \eta)|} \int_a^b \left[\int_0^{b-a} u^{\mathrm{Re}(\rho)(\gamma+k) - \mathrm{Re}(\eta) - 1} du \right] |\psi(t)|\, dt, \quad t > a.$$

Now interpreting the inner integral using term-by-term integration and taking into account (2.2), we at once arrive at the desired result in (2.1).

To prove Theorem-2, we know that the integral operator $G_{\rho,\eta,\gamma,\omega;a+}$ is bounded on L(a,b) by Theorem-1, so it is also bounded in the space C[a,b] of continuous function g on L(a,b) with a finite norm $\| g \|_C = \max\limits_{a \le x \le b} |g(x)|$

Using this concept and definition of integral operator $G_{\rho,\eta,\gamma,\omega;a+}$ in (1.8) and in view of (1.3) we have for any $x \in [a, b]$ and $\psi \in C[a, b]$,

$$\left\| \left(G_{\rho,\eta,\gamma,\omega;\,a+} \psi\right)(x) \right\| \leqq \int_a^x \left| (x-t)^{\rho\gamma - \eta - 1} \left[\sum_{k=0}^{\infty} \frac{(\gamma)_k \left[\omega(x-t)^{\rho}\right]^k}{k!\ \Gamma(\rho\gamma + \rho k - \eta)} \right] \psi(t) \right| dt$$

$$\leqq \| \psi \|_C \int_0^{b-a} \left| \sum_{k=0}^{\infty} u^{\mathrm{Re}(\rho)(\gamma+k) - \mathrm{Re}(\eta) - 1} du \, \frac{(\gamma)_k\, \omega^k}{k!\ \Gamma(\rho\gamma + \rho k - \eta)} \right|.$$

The integral on the right hand side is less than or equal to B^*, which is defined in (2.2).
This completes the proof of Theorem-2.

III. Compositions of the Operator $\left(G_{\rho,\eta,\gamma,\omega;a+\psi}\right)(x)$ and the Inversion Formula

Let $\rho, \alpha, \sigma, q, \eta, \gamma, \omega, \beta \in C, \left(\mathrm{Re}(q), \mathrm{Re}(\eta), \mathrm{Re}(\rho), \mathrm{Re}(\alpha) > 0\right)$ then the following results hold for $\psi \in L(a, b)$.

Result-1 $\left(G_{\rho,\eta,\gamma,\omega;\,a+} (t-a)^{\beta-1}\right)(x) = \Gamma(\beta)\, G_{\rho,\eta-\beta,\gamma}[\omega, (x-a)]$ 3.1

Result-2 $I_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+} \psi = G_{\rho,\eta-\alpha,\gamma,\omega;a+} \psi = G_{\rho,\eta,\gamma,\omega;a+}\, I_{a+}^{\alpha} \psi$ 3.2

Result-3 $D_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+} \psi = G_{\rho,\, \eta+\alpha,\gamma,\omega;\, a+} \psi$ 3.3

holds for any continuous function $\psi \in C[a, b]$.

Result-4 $G_{\rho,\eta,\gamma,\omega;a+} G_{\rho,q,\sigma,\omega\,;\,a+}\,\psi = G_{\rho,\eta+q,\gamma+\sigma,\omega;\,a+}\,\psi$ 3.4

Result-5 $G_{\rho,\eta,\gamma,\omega;a+}\, G_{\rho,q,-\gamma,\omega;a+}\,\psi = I_{a+}^{-(\eta+q)}\psi$ 3.5

Result-6 Let $G_{\rho,\eta,\gamma,\omega;a+}$ is invertible in the space $L(a,b)$ and for $\psi \in L(a,b)$,

$$\left(G_{\rho,\eta,\gamma,\omega;a+}\psi\right)(x) = f(x), \qquad a \le x \le b \tag{3.6}$$

Then $\left\{\left[G_{\rho,\eta,\gamma,\omega\,;a+}\right]^{-1} f\right\}(x) = D_{a+}^{-(\eta+q)}\left(G_{\rho,q-\gamma,\omega;a+}f\right)(x)$ 3.7

Proof of (3.1) To prove the result in (3.1), we denote its left hand side by Δ_5 i.e. $\Delta_5 = \left(G_{\rho,\eta,\gamma,\omega;a+}\,(t-a)^{\beta-1}\right)(x)$

Now using the definition of operator in (1.8) we have

$$\Delta_5 = \int_a^x G_{\rho,\eta,\gamma}\left[\omega,(x-t)\right](t-a)^{\beta-1}\,dt$$

with the help of definition in (1.3) and then changing the order of integration and summation we have

$$\Delta_5 = \sum_{n=0}^{\infty} \frac{(\gamma)_n\,\omega^n}{\Gamma(\rho\gamma+\rho n-\eta)\,n\,!} \int_a^x (t-a)^{\beta-1}(x-t)^{\rho\gamma+\rho n-\eta-1}\,dt$$

Evaluating the inner integral with the help of beta integral we have,

$$\Delta_5 = \Gamma(\beta)\,(x-a)^{\rho\gamma-(\eta-\beta)-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n\,\left[\omega(x-a)^{\rho}\right]^n}{n\,!\;\Gamma[\rho\gamma+\rho n-(\eta-\beta)]}\,.$$

On interpreting the resulting series with the help of (1.3), we at once arrive at the desired result in (3.1).

Proof of (3.2)

To prove the result in (3.2) we denote its left hand side by Δ_6 i.e.

$$\Delta_6 = \left(I_{a+}^{\alpha}\;G_{\rho,\eta,\gamma,\omega;a+}\psi\right)(x)\,.$$

On using the definition of the operator in (1.8) and applying Dirichlet formula for x>a, we have :

$$\Delta_6 = \int_a^x \left(\left\{I_{a+}^{\alpha}\left[G_{\rho,\eta,\gamma}(\omega,\tau)\right]\right\}(x-t)\right)\,\psi(t)\,dt.$$

Now on using the relation (1.6) we at once arrive at the desired result in (3.2) in accordance with the definition of operator in (1.8). The second relation of (3.2) is proved following similar lines as above.

Proof of (3.3)

To prove the result in (3.3), we denote its left hand side by Δ_7 i.e.

$$\Delta_7 = \left(D_{a+}^{\alpha}\;G_{\rho,\eta,\gamma,\omega;\,a+}\psi\right)(x)$$

on using the definition of fractional derivative D_{a+}^{α} in (3.2) we have,

$$\Delta_7 = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha}\;G_{\rho,\eta,\gamma,\omega;a+}\psi\right)(x)\,.$$

which in view of (3.2) takes the following form :

$$\Delta_7 = \left(\frac{d}{dx}\right)^n \left(G_{\rho,\eta-n+\alpha,\gamma,\omega;a+}\psi\right)(x)\,.$$

On applying the definitions in (1.8) and (1.4) we at once arrive at the desired result in (3.3) in accordance with the definition of the operator in (1.8).

Proof of (3.4)

To prove the result in (3.4), we denote its left hand side by Δ_8 i.e.

$$\Delta_8 = \left(G_{\rho,\eta,\gamma,\omega;a+}\,G_{\rho,q,\sigma,\omega;a+}\psi\right)(x)\,.$$

Now in view of the definition in (1.8) we have :

$$\Delta_8 = \int_a^x G_{\rho,\eta,\gamma}\left[\omega,(x-t)\right]\left(G_{\rho,q,\sigma,\omega;a+}\psi\right)(t)\,dt$$

$$= \int_a^x \left[G_{\rho,\eta,\gamma}\left[\omega,(x-t)\right] \int_a^t G_{\rho,q,\sigma}\left[\omega,(t-u)\right]\psi(u)\,du \right] dt$$

Now on using the definition of G-function in (3.3) and on changing the order of integration we have,

$$\Delta_8 = \int_a^t \left[\sum_{l_1,l_2=0}^{\infty} \frac{(\gamma)_{l_1}(\sigma)_{l_2}(\omega)^{l_1+l_2}}{l_1!\,l_2!\,\Gamma(\rho\gamma+\rho l_1-\eta)\,\Gamma(\rho\sigma+\rho l_2-q)} \right.$$

$$\left. .\int_a^x (x-t)^{\rho\gamma+\rho l_1-\eta-1}(t-u)^{\rho\sigma+\rho l_2-q-1}\,dt \right]\psi(u)\,du\,.$$

Now on evaluating the inner integral with the help of beta-integral we have

$$\Delta_8 = \int_a^t \left[\sum_{l_1,l_2=0}^{\infty} \frac{(\gamma)_{l_1}(\sigma)_{l_2}(\omega)^{l_1+l_2}(x-u)^{\rho(\gamma+\sigma)+\rho(l_1+l_2)-(\eta+q)-1}}{l_1!\,l_2!\,\Gamma\left[\rho(\gamma+\sigma)+\rho(l_1+l_2)-(\eta+q)\right]} \right]\psi(u)\,du\,,$$

$$\Delta_8 = \int_a^t (x-u)^{\rho(\gamma+\sigma)-(\eta+q)-1}\left[\sum_{l=0}^{\infty} \frac{(\gamma+\sigma)_l\left(\omega(x-u)^{\rho}\right)^l}{l!\,\Gamma\left[\rho(\gamma+\sigma)+\rho l-(\eta+q)\right]} \right]\psi(u)\,du$$

Now on interpreting the resulting series with the help of (3.2) and then in view of (1.8), we at once arrive at the desired result in (3.4).

Proof of (3.5)

To prove the result in (3.5), we denote its left hand side by Δ_9 i.e.

$$\Delta_9 = \left(G_{\rho,\eta,\gamma,\omega;a+}\,G_{\rho,q,-\gamma,\omega;a+}\psi\right)(x)$$

Now in view of the definition in (1.8) we have :

$$\Delta_9 = \int_a^x G_{\rho,\eta,\gamma}\left[\omega,(x-t)\right]\left(G_{\rho,q,-\gamma,\omega;a+}\psi\right)(t)\,dt$$

$$= \int_a^x \left[G_{\rho,\eta,\gamma}\left[\omega,(x-t)\right] \int_a^t G_{\rho,q,-\gamma}\left[\omega,(t-u)\right]\psi(u)\,du \right] dt\cdot$$

Now on using the definition of G-function in (1.3) and on changing the order of integration we have

$$\Delta_9 = \int_a^t \left[\sum_{l_1,l_2=0}^{\infty} \frac{(\gamma)_{l_1}(-\gamma)_{l_2}(\omega)^{l_1+l_2}}{l_1!\,l_2!\,\Gamma(\rho\gamma+\rho l_1-\eta)\,\Gamma(\rho(-\gamma)+\rho l_2-q)} \right.$$

$$\left. .\int_a^x (x-t)^{\rho\gamma+\rho l_1-\eta-1}(t-u)^{-\rho\gamma+\rho l_2-q-1}\,dt \right]\psi(u)\,du\,.$$

Now on evaluating the inner integral with the help of beta-integral we have

$$\Delta_9 = \int_a^t \left[\sum_{l_1,l_2=0}^{\infty} \frac{(\gamma)_{l_1}(-\gamma)_{l_2}(\omega)^{l_1+l_2}(x-u)^{\rho(l_1+l_2)-(\eta+q)-1}}{l_1!\,l_2!\,\Gamma\left[\rho(l_1+l_2)-(\eta+q)\right]} \right]\psi(u)\,du$$

which on making use of the series identify we have :

$$\Delta_9 = \frac{1}{\Gamma(-\eta-q)}\int_a^t (x-u)^{-(\eta+q)-1}\psi(u)\,du$$

Now with the help of the definition of the operator in (1.1), we at once arrive at the desired result in (3.5).

Proof of (3.6,3.7)

To prove the result in (3.7), let $\left(G_{\rho,\eta,\gamma,\omega\,;\,a+}\psi\right)(x) = f(x)$.

Now on operating $G_{\rho,q,-\gamma,\,\omega;a+}$ on both the sides we have

$$\left(G_{\rho,q,-\gamma,\omega;a+}\, G_{\rho,\eta,\gamma,\omega;a+}\psi\right)(x) = \left(G_{\rho,q,-\gamma,\omega;a+} f\right)(x)$$

which in view of (3.5) gives $\left(I_{a+}^{-(\eta+q)}\psi\right)(x) = \left(G_{\rho,q,-\gamma,\omega;a+} f\right)(x)$.

Now on operating $D_{a+}^{-(\eta+q)}$ on both the sides it gives :

$$\left(D_{a+}^{-(\eta+q)} I_{a+}^{-(\eta+q)}\psi\right)(x) = \left(D_{a+}^{-(\eta+q)}\, G_{\rho,q,-\gamma,\omega\,;\,a+} f\right)(x)$$

$$\psi(x) = \left(D_{a+}^{-(\eta+q)}\, G_{\rho,q,-\gamma,\omega;\,a+} f\right)(x)$$

which is the result in (3.7).

Now let $\left(\left[G_{\rho,\eta,\gamma,\omega\,;\,a+}\right]^{-1} f\right)(x) = \left(D_{a+}^{-(\eta+q)}\, G_{\rho,q,-\gamma,\omega;a+} f\right)(x)$

on operating $G_{\rho,\eta,\gamma,\omega;a+}$ both the sides, we have

$$\left(\left[G_{\rho,\eta,\gamma,\omega;a+}\right]^{-1}\left[G_{\rho,\eta,\gamma,\omega;\,a+}\right] f\right)(x) = \left(D_{a+}^{-(\eta+q)} G_{\rho,q,-\gamma,\omega;\,a+}\, G_{\rho,\eta,\gamma,\omega\,;\,a+} f\right)(x)$$

which in view of (3.5) gives

$$\left(\left[G_{\rho,\eta,\gamma,\omega\,;\,a+}\right]^{-1}\left[G_{\rho,\eta,\gamma,\omega\,;\,a+}\right] f\right)(x) = D_{a+}^{-(\eta+q)}\, I_{a+}^{-(\eta+q)}\, f(x)$$

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