On Absolute Weighted Mean $|A, \delta|_k$-Summability Of Orthogonal Series

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Generalizing the theorem of Kransniqi [On absolute weighted mean summability of orthogonal series, Slecuk J. Appl. Math. Vol. 12 (2011) pp 63-70], we have proved the following theorem which gives some interesting and new results.

If the series
\[
\sum_{n=1}^{\infty} \left( a_{n,n} \left\lfloor \frac{1}{k} \right\rfloor \sum_{j=0}^{\infty} |\hat{a}_{n,j}|^2 c_j^2 \right)^{\frac{k}{2}}.
\]
Converges for $1 \leq k \leq 2$ then the orthogonal series
\[
\sum_{n=1}^{\infty} c_n \psi_n(x)
\]
is summable $|A, \delta|_k$ almost every where,

Abstract: In this paper we prove the theorems on absolute weighted mean $|A, \delta|_k$-summability of orthogonal series. These theorems are generalize results of Kransniqi [1].

Keywords: Orthogonal Series, Nörlund Matrix, Summability.

I. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$ and Let $A = (a_{n,v})$ be a normal matrix, that is lower-semi matrix with non-zero entries. By $A_n(s)$ we denote the $A$-transform of the sequence $s = \{s_n\}$, i.e.
\[
A_n(s) = \sum_{v=0}^{\infty} a_{n,v} s_v.
\]
The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|A|_k$, $k \geq 1$ (Sarigöl [4]) if
\[
\sum_{n=0}^{\infty} |a_{n,n}|^{1/k} |A_n(s) - A_{n-1}(s)|^{k} < \infty
\]
And the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|A, \delta|_k$ $k \geq 1$, $\delta \geq 0$ if
\[
\sum_{n=0}^{\infty} |a_{n,n}|^{-\delta k+1-k} |\Delta A_n(s)|^{\delta k} < \infty
\]
where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$

In the special case when $A$ is a generalized Nörlund matrix $|A|_k$ summability is the same as $|N, p,q|_k$ summability (Sarigöl [5]).

By a generalized Nörlund matrix we mean one such that
\[
a_{n,v} = \frac{p_n - vq_n}{R_n} \quad \text{for} \quad 0 \leq v \leq n
\]
\[
a_{n,v} = O \quad \text{for} \quad v > n
\]
where for given sequences of positive real constant \( p = \{p_n\} \) and \( q = \{q_n\} \), the convolution \( R_n = (p * q)_n \) is defined by \[
(p * q)_n = \sum_{v=0}^{\infty} p_v q_{n-v} = \sum_{v=0}^{\infty} p_{n-v} q_v,
\]
where \((p * q)_n \neq 0\) for all \( n \), the generalized Nörlund transform of the sequence \( \{s_n\} \) is the sequence \( \{t^{p,q}_n(s)\} \) defined by \[
t^{p,q}_n(s) = \frac{1}{R_n} \sum_{m=0}^{\infty} p_{n-m} q_m s_m,
\]
and \( |A|_k \) summability reduces to the following definition:

The infinite series \( \sum_{n=0}^{\infty} a_n \) is absolutely summable \( |N,p,q|_k \) if the series
\[
\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \left( \frac{R_j}{R_n} - \frac{R_{j+1}}{R_{n+1}} \right)^2 \right) |t^{p,q}_n(s) - t^{p,q}_{n-1}(s)|^4 < \infty
\]
and we write \( \sum_{n=0}^{\infty} a_n \in N,p,q |_k \).

Let \( \{\psi(x)\} \) be an orthonormal system defined in the interval \((a,b)\). We assume that \( f(x) \in L^2(a,b) \) and
\[
f(z) \sim \sum_{n=0}^{\infty} c_n \psi_n(x)
\]
where \( c_n = \int_{a}^{b} f(x) \psi_n(x) dx, \ n = 0,1,2... \)
we write \( R_n = \sum_{v=0}^{\infty} p_v q_v, \ R_{n+1}^0 = 0, \ R_{n+1} = R_n \)
and \( P_n = (p * 1)_n = \sum_{v=0}^{\infty} p_v \) and \( Q_n = (1 * q)_n = \sum_{v=0}^{\infty} q_v \).

Regarding to \( |N,p,q|_k \equiv |N,p,q|_k \), summability of orthogonal series (1.1) the following two theorems are proved.

**Theorem 1.1** (Okuyama [3]) If the series
\[
\sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} \left( \frac{R_j}{R_n} - \frac{R_{j+1}}{R_{n+1}} \right)^2 \right) |c_j|^4 < \infty
\]
then the orthogonal series \( \sum c_n \psi_n(x) \) is summable \( |N,p,q|_k \) almost every where.

**Theorem 1.2** (Okuyama [3]) Let \( \{\Omega(n)\} \) be a positive sequence such that \( \{\Omega(n)/n\} \) is a non increasing sequence and the series \( \sum_{n=1}^{\infty} (n \Omega(n))^{-1} \) converges. Let \( \{p_n\} \) and \( \{q_n\} \) be non negative. If the series
\[
\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w^{(1)}(n) \]
converges then the orthogonal series \( \sum_{j=0}^{\infty} c_j \psi_j(x) \in N,p,q |_k \) almost everywhere, where \( w^{(1)}(n) \) is defined by
\[
w^{(1)}(j) = j^{-1} \sum_{n=j}^{\infty} n^2 \left( \frac{R_j}{R_n} - \frac{R_{j+1}}{R_{n+1}} \right)^2.
\]

The main purpose of this paper is studying of the \( |A,\delta|_k \) summability of the orthogonal series (1.1), for \( 1 \leq k \leq 2 \). Before starting the main result we introduce some further notations.
Given a normal matrix $A = a_{nv}$, we associate two lower semi matrices $\bar{A} = \bar{a}_{nv}$ and $\hat{A} = \hat{a}_{nv}$ as follows

$$\bar{a}_{nv} = \sum_{i=0}^{\infty} a_{nv}, \quad n, i = 0, 1, 2$$

and

$$\hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad \hat{a}_{0v} = \bar{a}_{0v}, \quad n = 1, 2, \ldots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations respectively.

Throughout this paper we denote by $\mathcal{K}_a$ a constant that depends only on $k$ and may be different in different relations.

### II. Main Results

We prove the following theorems:

**Theorem 2.1** If the series

$$\sum_{n=1}^{\infty} \left| a_{nv} \right| \left( \frac{1}{k} \right)^{1/2} \left( \sum_{j=0}^{n} \left| \hat{a}_{nv} \right|^2 \right)^{1/2}$$

Converges for $1 \leq k \leq 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n \psi_n(x)$$

is summable $|A, \delta|_k$ almost everywhere.

**Proof.** For the matrix transform $A_n(s)(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x)$ we have

$$A_n(s)(x) = \sum_{v=0}^{n} a_{nv} s_n(x)$$

$$= \sum_{v=0}^{n} a_{nv} \sum_{j=0}^{n} c_j \psi_j(x)$$

$$= \sum_{j=0}^{n} c_j \psi_j(x) \sum_{v=0}^{n} a_{nv}$$

$$= \sum_{j=0}^{n} \bar{a}_{nv} c_j \psi_j(x)$$

where $\sum_{j=0}^{n} c_j \psi_j(x)$ is the partial sum of order $v$ of the series (1.1)

Hence

$$\bar{A}A_n(s)(x) = \sum_{j=0}^{n} \bar{a}_{nv} c_j \psi_j(x) - \sum_{j=0}^{n-1} \left( \bar{a}_{nv} - \bar{a}_{n-1,v} \right) c_j \psi_j(x)$$

$$= \bar{a}_{nv} c_n \psi_n(x) + \sum_{j=0}^{n-1} \bar{a}_{n,j} c_j \psi_j(x)$$

$$= \hat{a}_{nv} c_n \psi_n(x) + \sum_{j=0}^{n-1} \hat{a}_{n,j} c_j \psi_j(x)$$

$$= \sum_{j=0}^{n} \hat{a}_{nv} c_j \psi_j(x)$$

Using the Hölder’s inequality and orthogonality to the latter equality we have
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\[
\int_{a}^{b} |\tilde{A}_{s}(s(x))|dx \leq (b-a)^{1-k} \left( \int_{a}^{b} |A_{s}(s(x)) - A_{s+1}(s(x))|^{2}dx \right)^{\frac{k}{2}}
\]

Thus the series
\[
\sum_{n=1}^{\infty} \left| a_{m} \right|^{1-k-dk} \int_{a}^{b} |\tilde{A}_{s}(s(x))|dx
\]

\[
\leq k \sum_{n=1}^{\infty} \left| n \right|^{2} \sum_{j=0}^{n} \left| \tilde{a}_{n,j} \right|^{2} |c_{j}|^{2}
\]

Converges by the assumption.

From this fact and since the function \( |\tilde{A}_{s}(s(x))| \) are non negative and by Lemma of Beppo-Levi [2] we have
\[
\sum_{n=1}^{\infty} \left| a_{m} \right|^{1-k-dk} |\tilde{A}_{s}(s(x))|^{k}
\]

Converges almost everywhere.

This completes the proof of theorem.

If we put
\[
H^{(k)}(A, \delta, j) = \frac{1}{j^{k-1}} \sum_{n=1}^{\infty} \left| n \right|^{2} \left( \sum_{j=0}^{n} \left| \tilde{a}_{n,j} \right|^{2} |c_{j}|^{2} \right)^{\frac{k}{2}}
\]

then the following theorem holds true.

**Theorem 2.2:** Let \( 1 \leq k \leq 2 \) and \( \{ \Omega(n) \} \) be a positive sequence such that \( \frac{\Omega(n)}{n} \) is a non decreasing sequence and the series \( \sum_{n=0}^{\infty} \frac{1}{n \Omega(n)} \) converges. If the following series \( \sum_{n=1}^{\infty} \left| c_{j} \right|^{2} \Omega^{(k)}(n) H^{k}(A, \delta, n) \)

converges, then the orthogonal series \( \sum_{n=0}^{\infty} c_{j} \nu_{n}(x) \in A, \delta \}_{k} \) almost everywhere, where \( H^{k}(A, \delta, n) \) is defined by (2.2)

**Proof:** Applying Hölder’s inequality to inequality (2.1) we get
\[
\sum_{n=1}^{\infty} \left| a_{m} \right|^{1-k-dk} \int_{a}^{b} |\tilde{A}_{s}(s(x))|dx \leq \sum_{n=1}^{\infty} \left| a_{m} \right|^{1-k-dk} \left[ \sum_{j=0}^{n} \left| \tilde{a}_{n,j} \right|^{2} |c_{j}|^{2} \right]^{\frac{k}{2}}
\]

\[
= k \sum_{n=1}^{\infty} \frac{1}{(n \Omega(n))^{\frac{2-k}{2}}} \left[ \sum_{j=0}^{n} \left| \tilde{a}_{n,j} \right|^{2} |c_{j}|^{2} \right]^{\frac{k}{2}}
\]

\[
\leq k \left( \sum_{n=1}^{\infty} \frac{1}{a_{n,n} \Omega(n)} \right)^{\frac{2-k}{2}} \left[ \sum_{n=1}^{\infty} \left| a_{m} \right|^{2} \left( \frac{n \Omega(n)}{n} \right)^{\frac{2-k}{2}} \sum_{j=0}^{n} \left| \tilde{a}_{n,j} \right|^{2} |c_{j}|^{2} \right]^{\frac{k}{2}}
\]

\[
\leq k \left( \sum_{j=1}^{\infty} \left| c_{j} \right|^{2} \sum_{n,j} \left| a_{n,n} \right|^{2} \left( \frac{n \Omega(n)}{n} \right)^{\frac{2-k}{2}} \left| \tilde{a}_{n,j} \right|^{2} \right)^{\frac{k}{2}}
\]

\[
\leq k \left( \sum_{j=1}^{\infty} \left| c_{j} \right|^{2} \left( \frac{\Omega(j)}{j} \right)^{\frac{2-k}{2}} \sum_{n,j} \left| n \right|^{2} \left| a_{m} \right|^{2} \left( \frac{n \Omega(n)}{n} \right)^{\frac{2-k}{2}} \left| \tilde{a}_{n,j} \right|^{2} \right)^{\frac{k}{2}}
\]

which is finite by virtue of the hypothesis of the theorem and completes the proof of the theorem.
References: