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On Absolute Weighted Mean $|A,\delta|_k$-Summability Of Orthogonal Series

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Generalizing the theorem of Kransniqi [On absolute weighted mean summability of orthogonal series, Slecuk J. Appl. Math. Vol. 12 (2011) pp 63-70], we have proved the following theorem which gives some interesting and new results.

If the series

$$\sum_{n=1}^{\infty}\left\{|a_{n,n}|^{2\left(\delta+\frac{1}{k}-1\right)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right\}^{\frac{k}{2}}.$$

Converges for $1 \le k \le 2$ then the orthogonal series

$$\sum_{n=1}^{\infty} c_n \psi_n(x)$$

is summable $|A,\delta|_k$ almost every where,

Abstract: *In this paper we prove the theorems on absolute weighted mean $|A,\delta|_k$-summability of orthogonal series. These theorems are generalize results of Kransniqi [1].*
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I. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$ and Let $A=(a_{n,v})$ be a normal matrix, that is lower-semi matrix with non-zero entries. By $A_n(s)$ we denote the A-transform of the sequence $s=\{s_n\}$, i.e.

$$A_n(s)=\sum_{v=0}^{\infty} a_{nv}s_v.$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|\mathrm{A}|_k$, $k \ge 1$ (Sarigöl [4]) if

$$\sum_{n=0}^{\infty}|a_{nn}|^{1-k}|A_n(s)-A_{n-1}(s)|^k<\infty$$

And the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|A,\delta|_k$ $k \ge 1$, $\delta \ge 0$ if

$$\sum_{n=0}^{\infty}|a_{nn}|^{-\delta k+1-k}|\bar{\Delta}A_n(s)|^k<\infty$$

where $\bar{\Delta}A_n(s)=A_n(s)-A_{n-1}(s)$

In the special case when A is a generalized Nörlund matrix $|\mathrm{A}|_k$ summability is the same as $|N,p,q|_k$ summability (Sarigöl [5]).

By a generalized Nörlund matrix we mean one such that

$$a_{nv}=\frac{p_n-vq_v}{R_n} \quad \text{for } 0 \le v \le n$$

$$a_{nv}=O \quad \text{for } v>n$$

where for given sequences of positive real constant $p=\{p_n\}$ and $q=\{q_n\}$, the convolution $R_n=(p*q)_n$ is defined by

$$(p*q)_n=\sum_{v=0}^{\infty}p_v q_{n-v}=\sum_{v=0}^{\infty}p_{v-v}q_v$$

where $(p*q)_n\neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}(s)\}$ defined by

$$t_n^{p,q}(s)=\frac{1}{R_n}\sum_{m=0}^{\infty}p_{n-m}q_m s_m$$

and $|A|_k$ summability reduces to the following definition:

The infinite series $\sum_{n-0}^{\infty}a_n$ is absolutely summable $|\mathrm{N,p,q}|_k$ $k\geq 1$ if the series

$$\sum_{n=0}^{\infty}\left(\frac{R_n}{q_n}\right)^{k-1}|t_n^{p,q}(s)-t_{n-1}^{p,q}(s)|^k<\infty$$

and we write

$$\sum_{n=0}^{\infty}a_n\in|N,\mathrm{p,q}|_k$$

Let $\{\psi(x)\}$ be an orthonormal system defined in the interval (a,b). We assume that $f(x)$ belongs to $L^2(a,b)$ and

$$f(z)\sim\sum_{n=0}^{\infty}c_n\psi_n f(x) \tag{1.1}$$

where $c_n=\int_a^b f(x)\psi_n(x)dx,\quad n=0,1,2...$

we write $R_n^j=\sum_{v=j}^{\infty}p_{n-v}q_v,\ R_n^{n+1}=0,\ R_n^0=R_n$

and $P_n=(p*1)_n=\sum_{v=0}^{\infty}p_v$ and $Q_n=(1*q)_n=\sum_{v=0}^{\infty}q_v$

Regarding to $|N,p,q|_1\equiv|N,p,q|$, summability of orthogonal series (1.1) the following two theorems are proved.

Theorem 1.1 (Okuyama [3]) If the series

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2|c_j|^2\right\}^{1/2}<\infty$$

then the orthogonal series $\Sigma c_n\psi_n(x)$ is summable $|N,p,q|$ almost every where.

Theorem 1.2 (Okuyama [3]) Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(\mathrm{n})/\mathrm{n}\}$ is a non increasing sequence and the series $\sum_{n=1}^{\infty}(n\Omega(n))^{-1}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non negative. If the series $\sum_{n=1}^{\infty}|c_n|^2\,\Omega(n)w^{(1)}(n)$ converges then the orthogonal series $\sum_{j=0}^{\infty}c_j\psi_j(x)\in|N,p,q|$ almost everywhere, where $w_{(n)}^{(1)}$ is defined by

$$w_{(j)}^{(1)}=j^{-1}\sum_{n=j}^{\infty}n^2\left(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\right)^2.$$

The main purpose of this paper is studying of the $|A,\delta|_k$ summability of the orthogonal series (1.1), for $1\leq k\leq 2$. Before starting the main result we introduce some further notations.

Given a normal matrix $A = a_{nv}$, we associates two lower semi matrices $\overline{A} = \overline{a}_{n,v}$ and $\hat{A} = \hat{a}_{n,v}$ as follows

$$\overline{a}_{n,v} = \sum_{i=v}^{\infty} a_{ni},\ n,i = 0,1,2$$

and $\hat{a}_{nv} = \overline{a}_{nv} - \overline{a}_{n-1,v}$, $\hat{a}_{00} = \overline{a}_{00}, n = 1,2,\ldots$.

It ma be noted that $\overline{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations resp.
Throughout this paper we denote by Ka constant that depends only on k and may be different in different relations.

II. Main Results

We prove the following theorems:
Theorem 2.1 If the series

$$\sum_{n=1}^{\infty}\left\{|a_{n,n}|^{2\left(\delta+\frac{1}{k}-1\right)}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right\}^{k/2}$$

Converges for $1 \le k \le 2$, then the orthogonal series

$$\sum_{n=0}^{\infty} c_n\psi_n(x)$$

is summable $|A,\delta|_k$ almost every where.

Proof. For the matrix transform $A_n(\mathrm{s})(\mathrm{x})$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} c_n\psi_n(x)$ we have

$$\begin{aligned}
A_n(s)(x) &= \sum_{v=0}^{n} a_{nv}s_v(x)\\
&= \sum_{v=0}^{n} a_{n,v}\sum_{j=0}^{v} c_j\psi_n(x)\\
&= \sum_{j=0}^{n} c_j\psi_n(x)\sum_{v=0}^{n} a_{nv}\\
&= \sum_{j=0}^{n} \overline{a}_{n,j}c_j\psi_n(x)
\end{aligned}$$

where $\sum_{j=0}^{v} c_j\psi_n(x)$ is the partial sum of order v of the series (1.1)
Hence

$$\begin{aligned}
\overline{\Delta}A_n(s)(x) &= \sum_{j=0}^{\infty}\overline{a}_{n,j}c_j\psi_j(x) - \sum_{j=0}^{n-1}\overline{a}_{n-1,j}c_j\psi_j(x)\\
&= \overline{a}_{n,\mathrm{n}}c_j\psi_n(x) + \sum_{j=0}^{n-1}(\overline{a}_{n,j} - \overline{a}_{n-1,j})c_j\psi_j(x)\\
&= \hat{a}_{n,\mathrm{n}}c_j\psi_n(x) + \sum_{j=0}^{n-1}\hat{a}_{n,j}c_j\psi_j(x)\\
&= \sum_{j=0}^{n}\hat{a}_{n,j}c_j\psi_j(x)
\end{aligned}$$

Using the Hölder's inequality and orthogonality to the latter equality we have

$$\int_a^b |\bar{\Delta}A_n(s)(x)|^k\, dx \le (b-a)^{1-\frac{k}{2}} \left(\int_a^b |A_n(s)(x) - A_{n-1}(s)(x)|^2\, dx\right)^{\frac{K}{2}}$$

$$= (b-a)^{1-\frac{k}{2}} \left(\int_a^b \left|\sum_{j=0}^{n} \hat{a}_{n,j} c_j \psi_j(x)\right|^2\right)^{\frac{K}{2}}$$

Thus the series

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k-\delta k} \int_a^b |\bar{\Delta}A_n(s)(x)|^k\, dx$$

$$\le k \sum_{n=1}^{\infty} \left\{ |a_{nn}|^{\frac{2}{k}-2\delta-2} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right\}^{k/2} \tag{2.1}$$

Converges by the assumption.

From this fact and since the function $|\bar{\Delta}A_n(s)(x)|$ are non negative and by Lemma of Beppo-Levi [2] we have

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k-\delta k} |\bar{\Delta}A_n(s)(x)|^k$$

Converges almost every where.

This completes the proof of theorem.

If we put

$$H^{(k)}(A;\delta,j) = \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} |na_{nn}|^{\frac{2}{k}-2-2\delta} |\hat{a}_{n,j}|^2 \tag{2.2}$$

then the following theorem holds true.

Theorem 2.2: Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\left\{\frac{\Omega(n)}{n}\right\}$ is a non decreasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty} |c_n|^2\, \Omega^{\frac{2}{k}-1}(n) H^k(A,\delta,n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \psi_n(x) \in |A,\delta|_k$ almost every where, where $H^k(A,\delta,n)$ is defined by (2.2)

Proof. Applying Hölder's inequality to inequality (2.1) we get

$$\sum_{n=1}^{\infty} |a_{nn}|^{1-k-\delta k} \int_a^b \left|\bar{\Delta}A_n(s)(x)\right|^k dx \le \sum_{n=1}^{\infty} |a_{nn}|^{1-k-\delta k} \left[\sum_{j=1}^{n} |\hat{a}_{n,j}|^2 |c_j|^2\right]^{k/2}$$

$$= k \sum_{n=1}^{\infty} \frac{1}{(n\Omega(n))^{\frac{2-k}{2}}} \left[|a_{nn}|^{\frac{2}{k}-2-2\delta} (n\Omega(n))^{\frac{2}{k}-1} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2}$$

$$\le k \left(\sum_{n=1}^{\infty} \frac{1}{a_{n,n}\Omega(n)} \right)^{\frac{2-k}{2}} \left[\sum_{n=1}^{\infty} |a_{nn}|^{\frac{2}{k}-2-2\delta} (n\Omega(n))^{\frac{2}{k}-1} \sum_{j=0}^{n} |\hat{a}_{n,j}|^2 |c_j|^2 \right]^{k/2}$$

$$\le k \left\{ \sum_{j=1}^{\infty} |c_j|^2 \sum_{n=j}^{\infty} |a_{n,n}|^{\frac{2}{k}-2-2\delta} (n\Omega(n)^{\frac{2}{k}-1} |\hat{a}_{n,j}|^2 \right\}^{k/2}$$

$$\le k \left\{ \sum_{j=1}^{\infty} |c_j|^2 \left(\frac{\Omega(j)}{j}\right)^{\frac{2}{k}-1} \sum_{n=j}^{\infty} n^{\frac{2}{k}-\delta} |na_{nn}|^{\frac{2}{k}-2-\delta} |\hat{a}_{n,j}|^2 \right\}^{k/2}$$

$$= k \left\{ \sum_{j=1}^{\infty} |c_j|^2\, \Omega^{\frac{2}{k}-1} j\, H^k(A,\delta,j) \right\}^{k/2}$$

which is finite by virtue of the hypothesis of the theorem and completes the proof of the theorem.

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