Pythagorean Triangle and Special Pyramidal Numbers

M. A. Gopalan¹, V. Sangeetha², Manju Somanath³

¹ Department of Mathematics, Srimathi Indira Gandhi College, Trichy-2, India
²,³ Department of Mathematics, National College, Trichy-1, India

Abstract: Patterns of Pythagorean triangle, where, in each of which either a leg or the hypotenuse is a pentagonal pyramidal number and centered hexagonal pyramidal number, in turn are presented.

Keywords: Pythagorean triangles, pentagonal pyramidal, centered hexagonal pyramidal.

I Introduction

The method of obtaining three non-zero integers α, β and γ under certain relations satisfying the equation \( \alpha^2 + \beta^2 = \gamma^2 \) has been a matter of interest to various mathematicians [1,2,3]. In [4-12], special Pythagorean problems are studied. In this communication, we present yet another interesting Pythagorean problem. That is, we search for patterns of Pythagorean triangles where in each of which, either a leg or the hypotenuse is represented by a pentagonal pyramidal number and centered hexagonal pyramidal number, in turn.

II Notation

\( P_n^m \) - m-gonal pyramidal number of rank n
\( CP_n^m \) - centered m-gonal pyramidal number of rank n
\( t_{m,n} \) - polygonal number of rank n.

III Method of Analysis

Let \((m,n,k)\) represent a triple of non-zero distinct positive integers such that \( m = (k+1)n \)

Let \( P(\alpha, \beta, \gamma) \) be the Pythagorean triangle whose generators are \( mn \). Consider

\[ \alpha = 2mn; \quad \beta = m^2 - n^2; \quad \gamma = m^2 + n^2. \]

It is observed that, for suitable choices of \( n \), either a leg or hypotenuse of the Pythagorean triangle \( P \) is represented by a pentagonal pyramidal number and centered hexagonal pyramidal number, in turn. Different choices of \( n \) along with the corresponding sides of the Pythagorean triangle are illustrated below.

Choice 3.1

Let \( n = 4k + 3 \).

The corresponding sides of the Pythagorean triangle are

\[ \alpha = 32k^3 + 80k^2 + 66k + 18 \]
\[ \beta = 16k^4 + 56k^3 + 57k^2 + 18k \]
\[ \gamma = 16k^4 + 56k^3 + 89k^2 + 66k + 18 \]

Note that \( \alpha = P_n^5 \)

Choice 3.2

Let \( n = 2k^2 + 4k + 3 \)

The corresponding sides of the Pythagorean triangle are

\[ \alpha = 8k^5 + 40k^4 + 88k^3 + 104k^2 + 66k + 18 \]
\[ \beta = 4k^6 + 24k^5 + 60k^4 + 80k^3 + 57k^2 + 18k \]
\[ \gamma = 4k^6 + 24k^5 + 68k^4 + 112k^3 + 113k^2 + 66k + 18 \]

Note that \( \gamma = P_n^5 \)

Note

It is worth mentioning here that, for the following two choices of \( m,n \) given by (i) \( n = 4k, m = k(n + 1) \) and (ii) \( n = 2k^3 - 3, m = kn \) the sides \( \alpha \) and \( \beta \) represent \( P_n^5 \) respectively.

Choice 3.3

Let \( n = 2(k + 1) \)

The corresponding sides of the Pythagorean triangle are

\[ \alpha = 8k^3 + 24k^2 + 24k + 8 \]
\[ \beta = 4k^4 + 16k^3 + 20k^2 + 8k \]
\[ \gamma = 4k^4 + 16k^3 + 28k^2 + 24k + 8 \]

Note that \( \alpha = CP_n^5 \)
Choice 3.4
Let \( n = k(k + 2) \)
The corresponding sides of the Pythagorean triangle are
\[
\alpha = 2k^5 + 10k^4 + 16k^3 + 8k^2 \\
\beta = k^6 + 6k^5 + 12k^4 + 8k^3 \\
\gamma = k^6 + 6k^5 + 14k^4 + 16k^3 + 8k^2
\]
Note that \( \beta = CP_n^6 \)

Choice 3.5
Let \( n = k^2 + 2k + 2 \)
The corresponding sides of the Pythagorean triangle are
\[
\alpha = 2k^5 + 10k^4 + 24k^3 + 32k^2 + 24k + 8 \\
\beta = k^6 + 6k^5 + 16k^4 + 24k^3 + 20k^2 + 8k \\
\gamma = k^6 + 6k^5 + 18k^4 + 32k^3 + 36k^2 + 24k + 8
\]
Note that \( \gamma = CP_n^6 \).

Properties
(1) \( 3(\gamma - \beta) \) is a Nasty Number.
(2) \( \frac{\beta}{\gamma} \) is a biquadratic integer.
(3) \( \frac{\alpha}{\beta} \) is a perfect square.
(4) \( \frac{\gamma}{\beta} = \frac{CP_{n+1}^3}{k^2 + 2k + 3} \)
(5) \( \alpha \) is a perfect square when \( k = 2p^2 - 1 \)
(6) \( 6(\gamma - \alpha) \) is a Nasty number.
(7) \( \frac{\gamma}{\alpha} \) is a biquadratic integer.
(8) \( \frac{3\gamma}{\alpha} = \frac{CP_{n+1}^3}{k^2} \)

IV Conclusion
One may search for other patterns of Pythagorean triangles, where, in each of which either a leg or the hypotenuse is represented by other polygonal and pyramidal numbers.

References
[13]. M.A.Gopalan, V.Sangeetha and Manju Somanath, Pythagorean Triangle and Pentagonal Number, Accepted for publication in Cayley Journal of Mathematics.