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The proof of Riemann Hypothesis

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The condition for which $(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z})^2+(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})^2=0$ **where** Z **is a complex number reveals those points** Z **for which the functions** $(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z})+i(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})$ **and** $(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z})-i(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})$ **have zeroes.Finally,by direct analysis we can find zeroes of Riemann zeta funtion .**

I. Introduction

1.1 Riemann hypothesis

Theorem 1 (Riemann hypothesis) *All non-trival zeroes of Riemann zeta function defined by* $\zeta(Z)=\sum_{n=1}^{\infty}\frac{1}{n^Z}$ *where* Z *is a complex number ,lie on the line* $Z=(\frac{1}{2}+iy)$.

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Since $(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})=\frac{1}{2^Z}[(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z})+(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})]$, Therefore,

$$(2^Z-1)(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})=(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z}) \qquad (1)$$

Consequently, $(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^Z})^2+(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})^2=0$

Implies, $[(2^Z)^2-(2)(2^Z)+2](\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})^2=0$ Implies,either $(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})=0$ or $[(2^Z)^2-(2)(2^Z)+2]=0$ or Both. Let us assume first $(\sum_{n=1}^{\infty}\frac{1}{(2n)^Z})\neq 0$ Then, $2^Z=1\pm i$ Implies, $2^Z=\sqrt{2}(e)^{\frac{(8k\pm1)(\pi)(i)}{4}}$, Where, k, is any positive integer including zero. Implies, Z lie on the line $Z=\frac{1}{2}+[\frac{(8k\pm1)(\pi)(i)}{(4\ln 2)}]$ When,any point Z_1 liesonthe line $Z=\frac{1}{2}+[\frac{(8k+1)(\pi)(i)}{(4\ln 2)}]$ It follows from eq $(1.2.1.1)$,

$$2^{(}z_1)=1+i$$

$$(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)})-(i)(\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_1)})=0 \qquad (2)$$

Similarly,If any point Z_2 lies on the line $Z=\frac{1}{2}+[\frac{(8k-1)(\pi)(i)}{(4\ln 2)}]$ Then,it follows from eq $(1.2.1.1)$,

$$2^{(}z_2) = 1 - i$$

Consequently,

$$(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_2)}) + i(\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_2)}) = 0 \qquad (3)$$

But,We want something more.We wish to show that

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If any point

z_1

lies on the line

$$Z = (\frac{1}{2} + [\frac{(8k+1)(\pi)(i)}{(4\ln 2)}])$$

and another point

z_2

lies on the line

$$Z = (\frac{1}{2} + [\frac{(8k-1)(\pi)(i)}{(4\ln 2)}])$$

Then,modulas of $\left|\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}\right| = \left|\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_2)}\right| = \sigma$,where σ tends to zero.

But before showing this, let us show that four common values of all infinitely many valued function

$(e)^{[}\frac{(i)(\theta)(\ln p)}{\ln 2}]$ are (-1) ,1 and $\pm i$ where P is any odd prime taken arbitrarily.

Since

$$\frac{1}{(e)^t - 1} + \frac{1}{2} =$$

$$\frac{1}{2}(\frac{(e)^t + 1}{(e)^t - 1})$$

$$= \frac{(i)}{2}\cot(\frac{1}{2}it)$$

$$= \frac{1}{t} + (2t)\sum_{n=1}^{\infty}\frac{1}{t^2 + 4n^2(\pi)^2}$$

If $t = 2(\pi)(k)(i) + \frac{(\ln p)(i)(\theta)}{\ln 2}$

where k is any positive integerand $k \in (0,1,2,\ldots)$,

then as k tends to ∞ ,it follows, $\frac{1}{(e)^{[}2(\pi)(k)i + \frac{(\ln p)(i)(\theta)}{\ln 2}] - 1} + \frac{1}{2} = 0$ Consequently,

$$(e)^{[}2(\pi)(k)(i) + \frac{(\ln p)(i)(\theta)}{\ln 2}] = -1 \qquad (4)$$

Since, $(e)^{2(\pi)(k)i} = 1$ So it follows from eqution $1.3.2.1$ that one of the values of

$$(e)^{\frac{(i)(\theta)(\ln p)}{\ln 2}} = -1 \qquad (5)$$

Obviously then,chooseing suitable θ we can arrive at the values $\pm i$ and 1. Since the matter plays a key role in what follows, an example will not be out of place here.Let us consider the many valued function

$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}}$. Taking the value of $(\ln 3)$ and $(\ln 2)$ upto 9 decimal and using **De Moivre's Theorem** we find

$$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}} = (e)^{\frac{(1.098612289)(\pi)(i)}{(4)(0.69314718)}} = (e)^{\frac{(1.584962502)(\pi)(i)}{4}} = (e)^{\frac{(2-0.415037498)(\pi)(i)}{4}} = (i)(e)^{\frac{(103759374)(\pi)(-i)}{1000000000}} = (i)(e)^{\frac{(2n+103759374)(\pi)(-i)}{1000000000}}$$

Obviously then,when $n = 448120313$,

$$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}} = (-i)$$

When $n = 198120313$,then

$$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}} = 1$$

when $n = 948120313$,then

$$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}} = i$$

when $n = 698120313$,then

$$(e)^{\frac{(\ln 3)(\pi)(i)}{4\ln 2}} = -1$$

Actually,among infinitely many values,these four values are common to all $(e)^{\frac{(\ln P)(\pi)(i)}{4\ln 2}}$, where p is any odd prime taken arbitrarily. Now since,

$$\left(\frac{1}{1-\frac{1}{p^z}}\right) = \frac{p^z}{P^z - 1}$$

So,the function $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^Z}$ may be regarded as the rational function of two functions $\psi(z)$ and $\phi(z)$

i.e

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^Z} = \frac{\psi(z)}{\phi(z)}$$

where, $z = x + iy$, p_n denotes the n th prime and

$$\psi(z) = \left[\prod_{n=2} p_n^{x} (e)^{(\ln p_n)(y)(i)}\right]$$

and

$$\varphi(z) = \left[\prod_{n=2} p_n^{x} (e)^{(\ln p_n)(y)(i)}\right]^{-1}$$

So, it is not necessary that the branch of multivalued functions $\psi(z)$ and $\phi(z)$ have to be same all the time,In other words the value of $(e)^{(\ln p_n)(y)(i)}$ may be different for $\psi(z)$ and $\phi(z)$, If all

$$(e)^{\frac{(\pi)(i)(\ln p)}{4\ln 2}} = \pm(i) \qquad (6)$$

and if p_n denotes the n th prime then

$$\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_1)}}\right) = \prod_{n=2}^{\infty} \frac{1}{(1-\frac{1}{p_n^{Z_1}})} = \prod_{n=2}^{\infty} \frac{1}{(1+\frac{\pm i}{\sqrt{p_n}})} \qquad (7)$$

and

$$\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_2)}}\right) = \prod_{n=2}^{\infty} \frac{1}{(1-\frac{1}{p_n^{Z_2}})} = \prod_{n=2}^{\infty} \frac{1}{(1-\frac{\pm i}{\sqrt{p_n}})} \qquad (8)$$

Obviously then,

$$\left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_1)}}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_2)}}\right| = \prod_{n=2}^{\infty} \frac{1}{\sqrt{(1+\frac{1}{p_n})}} \qquad (9)$$

Since,

$$\prod_{n=2}^{\infty} \frac{1}{(1+\frac{1}{p_n})} \qquad (10)$$

$$\times\prod_{n=2}^{\infty}\frac{1}{(1-\frac{1}{p_n})} \qquad (11)$$

$$=\prod_{n=2}^{\infty}\frac{1}{(1-\frac{1}{p_n^2})} \qquad (12)$$

$$=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} \qquad (13)$$

$$=\frac{(\pi)^2}{8} \qquad (14)$$

And

$$\prod_{n=2}^{\infty}\frac{1}{(1-\frac{1}{p_n})}=\frac{(\ln n)-\Upsilon}{2} \qquad (15)$$

Where Υ is EULER'S CONSTANT. Obviously then, from eqations $(1.3.2{,}6)$, $(1.3.2.7)$, $(1.3.2.8)$, $(1.3.2.11)$ and $(1.3.2.12)$ it follows,

$$\prod_{n=2}^{\infty}\frac{1}{\sqrt{(1+\frac{1}{p_n})}}= \frac{\pi}{2\sqrt{(\ln n)-\Upsilon}} \qquad (16)$$

Consequently,when n tends to ∞ then lnn $\rightarrow \infty$ so $\frac{\pi}{2\sqrt{(\ln n)-\Upsilon}} \rightarrow \sigma$, where σ tends to zero. Therefore, from equations $(1.3.2.6)$ and $(1.3.2.13)$ it is clear that

$$\left|\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}\right| = \left|\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_2)}\right| = \sigma \qquad (17)$$

If all

$$e^{\frac{(\ln p)(\pi)(i)}{4\ln 2}}=-1 \qquad (18)$$

then since,

$$\left(\frac{1+\frac{1}{\sqrt{p}}}{\sqrt{(1+\frac{1}{P})}}\right)^2 = 1+\frac{2\sqrt{p}}{P+1} \qquad (19)$$

So,

$$\prod_{n=2}^{\infty} \frac{1}{\sqrt{(1+\frac{1}{P_n})}} = \sigma > \qquad (20)$$

$$\prod_{n=2}^{\infty} \frac{1}{1+\frac{1}{\sqrt{P_n}}}$$

When all $e^{\frac{(\ln P)(\pi)(i)}{4\ln 2}} = 1$ then dividing equation $(1.3.2.12)$ by inequality $(1.3.2.17)$ it follows

$$\prod_{n=2}^{\infty} \frac{1}{1-\frac{1}{\sqrt{P_n}}} > \frac{(\ln n) - \Upsilon}{2}$$

In otherwords, $\left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_1)}}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_2)}}\right| \to \infty$

But if in this case,the value of $(e)^{\frac{(\ln p)(\pi)(i)}{4\ln 2}}$ forthe function $\psi(z)$ be taken as $+1$ for the points Z_1 or Z_2 and -1 for the function $\phi(z)$,then it can be proved easily that $\left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_1)}}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_2)}}\right| \to \sigma$ where σ tends to zero.In otherwords, the function $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^Z}$ is convergent on the real axis. So far we have considered those cases when all $e^{\frac{(lnP)(\pi)(i)}{4\ln 2}}$ is $\pm i$ or (-1) , but it may happens that for some finite number of primes(we denote any such primeby p_d) the value of $e^{\frac{(\ln p_d)(\pi)(i)}{4\ln 2}}$ is different from the rest. But since $\prod_{d=d_1}^{d=d_k}(\frac{1}{1-\frac{1}{(p_d)^{(Z_1)}}})$ or $\prod_{d=d_1}^{d=d_k}(\frac{1}{1-\frac{1}{(p_d)^{(Z_2)}}})$ is bounded ,so it can be proved easily that $\left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_1)}}\right| = \left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{(z_2)}}\right| \to \sigma$ This completes proof of our assertion.

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It is clear from our above discussion that if $Z = \frac{1}{t} + i\theta$, where $0 < (\frac{1}{t}) < 1$ then $\zeta(Z)$ is convergent as because

$$|\zeta(Z)| \le \left(\left|\sum_{n=1}^{\infty} \frac{1}{(2n)^Z}\right| + \left|\sum_{n=1}^{\infty} \frac{1}{(2n-1)^Z}\right|\right) \le (m\sigma)$$

where m is some positive finite real number. Consequently, $\zeta(Z)$ is analytic on the right half plane i,e $R(z) > 0$. We have already told that Since,

$$(\frac{1}{1-\frac{1}{P^Z}})=(\frac{P^Z}{P^Z-1})$$

there is a possibility that in case of odd prime P, forany of $(\frac{P^Z}{P^Z-1})$ the value of (P^Z) of the numerator may be different from that of the denominator,as because $e^{(\ln p)(i)(\theta)}$ is infinitely many-valued. Let us explore such possibility for the point z_1 or z_2. Suppose,for any factor of $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}$, for the $P^{(}z_1)$ of the numerator,the value of $e^{\frac{(\ln p)(\pi)(i))}{4\ln 2}}=-1$ but for the $P^{(}z_1)$ ofthe denominator,the value of $e^{\frac{(\ln P)(\pi)(i)}{4\ln 2}}=(\pm i)$ Then,

$$\frac{P^{(}z_1)}{P^{(}z_1)-1}=\frac{\pm i}{1+\frac{\pm i}{\sqrt{P}}} \quad (21)$$

Consequently,

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}=(\pm i)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)} \quad (22)$$

Consequently, When $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}=-i\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}$, then for equation $(1.2.1.2)$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}=-\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_1)} \quad (23)$$

Consequently,

$$\zeta(z_1)=0 \quad (24)$$

When $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}=+i\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}$, then for equation $(1.2.1.2)$

$$(\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}-\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_1)})=0 \quad (25)$$

In otherwords, Dirichlet eta function becomes zero. Arguments are almost similar for z_2. On the otherhand,if none of the value of $P^{(}z_1)$ or $P^{(}z_2)$ of the numerator is different from that of the denominator,then for z_1, $\zeta(z_1)\neq 0$ but $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_1)}-i\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_1)}=0$ and for z_2, $\zeta(z_2)\neq 0$ but $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^{(}z_2)}+i\sum_{n=1}^{\infty}\frac{1}{(2n)^{(}z_2)}=0$ Thus all non-trival roots of Riemann zeta function as well as roots of Dirichlet eta function lie on the line $Z=\frac{1}{2}+[\frac{(8k\pm 1)(\pi)(i)}{4\ln 2}]$ and this completes the proof of Riemann Hypothesis.

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