IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 7, Issue 4 (Jul. - Aug. 2013), PP 01-03
www.iosrjournals.org

Finding Numbers Satisfying The Condition $A^n+B^n=C^n$.

T. Unnikrishnan
Academic Consultant, Department of Statistics, Kerala Veterinary and Animal Sciences University

Abstract: *Here an attempt is made to find positive numbers a,b and c such that $a^n+b^n=c^n$. Two cases were considered. a^n+b^n is not divisible by $(a+b)^2$ and a^n+b^n is divisible by $(a+b)^2$. From this a solution for the case n=3 is obtained as $(9+\sqrt{5})^3 + (9-\sqrt{5})^3 = 12^3$. A condition for finding such numbers for any n is also reached.*

Key words: *Fermat's Last Theorem, Number Theory, Mathematics, Diophantine equation, Pythagorean triples*

I. Introduction

Fermat's last theorem states that "$x^n+y^n=z^n$ has no nonzero integer solution for x,y,z when n>2". Pierre de Fermat wrote "I have discovered a truly wonderful proof which this margin is too small to contain". He almost certainly wrote the marginal note around 1630, when he first studied Diaphantus "Arithmetica". Unfortunately no proof has been found and his theorem remained unsolved as one of the most baffling unresolved problems of mathematics for over three and a half centuries. This theorem has a long history of inducing mathematicians to develop more and more theories to solve it. Volumes and volumes of higher mathematics have been created in an effort to prove or disprove the assertion. It has the classic ingredients of a problem to catch the imagination of a wide public – "a simple statement widely understood but a proof which defeats greatest intellects". In 1955 Yutaka Thaniyama asked some questions about elliptic curves. Further work by Weil & Shimura produced a conjucture now known as Shimura-Thaniyama-Weil conjucture. Further work by other mathematicians showed that a counter example to Fermat's last theorem would provide a counter example to Shimura-Thaniyama-Weil conjucture. During 21-23, June 1993 Andrew wiles, a British mathematician working at Princeton in U.S.A. gave a series of three lectures at Isaac Newton Institute for Mathematical Sciences in Cambridge. On Wednesday 23/06/93 at around 10:30 A.M. Wiles announced his proof of Fermat's last theorem as a corollary to his main results. This however is not the end of the story. On 4 December 1993 Andrew wiles made a statement in view of his speculation & withdrew his claims to have a proof. On 6 October Wiles sent the new proof to three colleagues including Faltings. Taylor lectured at the British Mathematical Colloquium in Edinburgh in April 1995 and the recently accepted proof of Wiles contains over 100 pages. No proof of the complexity of this can easily be guaranteed to be correct.

II. Results

To find numbers satisfying $x^n+y^n=z^n$, it need only to consider odd prime n only. Consider the case in which x,y,z are prime to each other and n as an odd prime. If $a^n+b^n=c^n$, where a,b,c are integers such that a+b is positive, then c can be considered as below

$$\frac{c}{a+b}(a+b) = \frac{d}{e}(a+b),$$ where d and e are prime to each other.

Surely a+b should be a multiple of e

Let $a+b = ke^m$, where k and e are positive integers such that k is not divisible by e.

$a^n+b^n = (a+b)(a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots\dots\dots\dots-ab^{n-2} + b^{n-1})$,if n>2 is odd.

Now consider the two cases a^n+b^n is divisible by $(a+b)^2$ or not.

Case 1 : If a^n+b^n is not divisible by $(a+b)^2$

Here $a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots\dots\dots\dots-ab^{n-2} + b^{n-1}$ will not be divisible by a+b.

Hence $(d^n/e^n)\,(ke^m)^{n-1}$ will not be divisible by ke^m.

Here mn-m-n should be less than m.

i.e, $n < 2m/(m-1)$ which is ≤ 4, if $m \geq 2$.

Since the case n=3 had proved early for positive integers a and b , $a^n+b^n \neq c^n$, if $m \geq 2$.

when $1 \leq m < 2$, $a+b = k_1e^{1+(r/s)} = ke$, where $k = k_1e^{(r/s)}$

The **case 3, a+b =ke** will be considered in general later.

Case 2 : If a^n+b^n is divisible by $(a+b)^2$

In this case $a^{n-1}-a^{n-2}b+a^{n-3}b^2-\dots\dots\dots\dots-ab^{n-2} + b^{n-1}$ will be divisible by (a+b).

Since $a \equiv -b(\text{mod}(a+b)), b \equiv -a(\text{mod}(a+b))$ & $a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots-ab^{n-2}+b^{n-1} \equiv 0(\text{mod}(a+b))$, $na^{n-1} \equiv 0(\text{mod}(a+b))$ and $nb^{n-1} \equiv 0(\text{mod}(a+b))$.
But as a and b are prime to each other a+b should be either 1 or n.
case 2:(a), a+b = 1 can be proved as corollary to the **case 3** which will be discussed later.
Case 2:(b), a+b = n.
Here replace b by n-a in $a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots-ab^{n-2}+b^{n-1} = (d^n/e^n)\,n^{n-1}$.
$\therefore ({}^nC_1)a^{n-1} - ({}^nC_2)\,a^{n-2}n + ({}^nC_3)a^{n-3}n^2 - \ldots\ldots - ({}^nC_{n-1})an^{n-2} + n^{n-1} = (d^n/e^n)\,n^{n-1}$.

Here if e>1 , LHS is an integer while RHS is not. It is impossible.
If e=1, $a^{n-1} \equiv 0(\text{mod } n)$, if n>2, which will contradict the assumption that a and b are co-prime pairs as a+b = n.

Thus in this case $a^n+b^n \neq c^n$.

Case 3 :Let a+b = ke.
Replacing b by ke-a in $a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots-ab^{n-2}+b^{n-1} = (d^n/e^n)(ke)^{n-1} == (d^n/e)\,k^{n-1}$, the equation becomes,
$na^{n-1}-({}^nC_2)a^{n-2}ke + \ldots\ldots -({}^nC_{n-1})\,a(ke)^{n-2} + (ke)^{n-1} = d^nk^{n-1}/e$ **…. ….. (1)**
Hence $na^{n-1} - d^nk^{n-1}/e \equiv 0(\text{Mod }(ke))$
Here if k^{n-1}/e is an integer larger than 1, any divisor of it will divide na^{n-1} also and the divisor will be n.
Hence k^{n-1}/e is either 1 or n.

When $k^{n-1}/e = 1$, From (1) we see that $d^n \equiv 1(\text{Mod } n)$

Since in this case $k \not\equiv 0(\text{Mod } n)$, $k^{n-1} \equiv 1(\text{Mod } n)$, i.e, $e \equiv 1(\text{Mod } n)$.
Thus in this case $d \equiv 1(\text{Mod } n)$ and $e \equiv 1(\text{Mod } n)$.
Also when $k^{n-1}/e = 1$, $a + b = ke = k^n$

$\therefore$ **a + b = 1** is a special case of this.
But as a+b, c-a and c-b cannot be 1 simultaneously, $a^n+b^n \neq c^n$, in this case

When $k^{n-1}/e = n$, from(1) it can be sen that, $d^n \equiv 1(\text{Mod } n)$. Now as $k^{n-1}/e = n$, $k \equiv 0(\text{Mod } n)$ & so $e \equiv 0(\text{Mod } n)$.
Thus in this case $d \equiv 1(\text{Mod} n)$ and $e \equiv 0(\text{Mod } n)$.
Also if $k^{n-1}/e = n$, c is a multiple of n and hence a & b will not be a multiple of n.

Now consider the case $x, y, z \in N$ and $x^n + y^n = z^n$

Let x < y < z and x, y, z are prime to each other.
We have $x^n + y^n = z^n = z^n (x+y)^n/(x+y)^n = (d_1/e_1)^n(x+y)^n$, Where d_1, e_1 are co-prime pairs.
$z^n - x^n = y^n = y^n (z-x)^n/(z-x)^n = (d_2/e_2)^n(z-x)^n$, Where d_2, e_2 are co-prime pairs.
$z^n - y^n = x^n = x^n (z-y)^n/(z-y)^n = (d_3/e_3)^n(z-y)^n$, Where d_3, e_3 are co-prime pairs.

From these equations, $x + y = k_1e_1^{m1}$, $z - x = k_2e_2^{m2}$, $z - y = k_3e_3^{m3}$.
Now from cases 1,2 and 3, it can be seen that $m_1 = m_2 = m_3 = 1$ and
$k_1^{n-1}/e_1 = 1$or n, $k_2^{n-1}/e_2 = 1$ or n, $k_3^{n-1}/e_3 = 1$ or n.

Here $x + y = k_1e_1$, $z - x = k_2e_2$, $z - y = k_3e_3$ and $z = d_1k_1$, $y = d_2k_2$, $x = d_3k_3$.

If x, y, z are not a multiple of n, $k_1^{n-1}/e_1 = k_2^{n-1}/e_2 = k_3^{n-1}/e_3 = 1$.
If one and only one of x, y, z is a multiple of n, with out loss of generality assume that z is a multiple of n.
Hence $k_1^{n-1}/e_1 = n$ & $k_2^{n-1}/e_2 = k_3^{n-1}/e_3 = 1$.

Consider the case $k_1^{n-1}/e_1 = k_2^{n-1}/e_2 = k_3^{n-1}/e_3 = 1$. Here if $k_1 = k_2 + k_3$, $k_1^n = (k_2 + k_3)^n$

$$\Rightarrow k_1^n - (k_2^n + k_3^n) = ({}^nC_1)\,k_2^{n-1}k_3 + \ldots\ldots + (({}^nC_{n-1})\,k_2k_3^{n-1}$$
$$\Rightarrow (x + y) - (z - x + z - y) = k_2k_3\,(({}^nC_1)\,k_2^{n-2} + \ldots\ldots + (({}^nC_{n-1}))\,k_3^{n-2})$$
$$\Rightarrow 2k_1h_1n = 2k_2h_2n = 2k_3h_3n = k_2k_3\,(({}^nC_1)k_2^{n-2} + \ldots\ldots + (({}^nC_{n-1}))k_3^{n-2})$$

Here LHS is even. So RHS should be even. But both k_2 or k_3 cannot be even. For this both $d_2k_2 = y$ and $d_3k_3 = x$ will be even, which is impossible.

If one of them is even and the other is odd then cancelling the even k_i by equating the corresponding LHS equation to RHS, a contradiction is arrived.

If both are odd, then $k_1=k_2 + k_3$ will be even, then as $k_3 = k_1 – k_2$,
$k_1^n-(k_2^n+k_3^n) = ((^nC_1)k_1^{n-1}k_2 – \ldots\ldots\ldots.-((^nC_{n-1}) k_1k_2^{n-2}$.
$\Rightarrow 2 h_1n = k_2[((^nC_1) k_1^{n-2} -\ldots\ldots\ldots\ldots\ldots-((^nC_{n-1})k_2^{n-2}]$.
Here as k_1 is even, a similar contradiction is arrived.
Thus in this case if x,y,z are not a multiple of n, $x^n+y^n \neq z^n$.
Here from case 3 it can be seen that $e_i \equiv$ (Mod n), $d_i \equiv 1$(Mod n), i =1, 2, 3.

Let $d_1=g_1n+1$, $d_2=g_2n+1$, $d_3=g_3n+1$, $e_1=(g_1 + h_1)n+1$, $e_2=(g_2 - h_2)n+1$, $e_3=(g_3-h_3)n+1$, where g_1, g_2, g_3, h_1, h_2, h_3 are all positive integers.

Now as $x < y < z$, $z – y < z – x < x + y$ i.e, $k_3^n < k_2^n < k_1^n$.
Hence $k_3 < k_2< k_1$

But (x + y) - z = y - (z - x) = x - (z - y).
i.e, $k_1e_1 – d_1k_1 = d_2k_2 - k_2e_2 = d_3k_3 -k_3e_3$.
i.e, $k_1h_1 = k_2h_2 = k_3h_3$.

Thus as $k_3 < k_2 < k_1$, $h_1< h_2< h_3$.

Now $x = k_3^n + nk_1h_1$, $y = k_2^n + nk_1h_1$, $z = k_1^n - nk_1h_1$

Consider the case $k_1^{n-1}/e_1 = n$, $k_2^{n-1}/e_2 = k_3^{n-1}/e_3 = 1$.

From case 3 we have $d_1 = g_1n+1$, $d_2 = g_2n+1$, $d_3 = g_3n+1$,
$e_1 = (g_1 +h_1)n$, $_2 = (g_2-h_2)n+1$, $e_3 = (g_3-h_3)n+1$, where gi and h_i are positive integers.

Now y-(z-x) = x-(z-y)
$\Rightarrow d_2k_2 – k_2e_2 = d_3k_3 – k_2e_3$
$\Rightarrow k_2h_2 = k_3h_3$.

In this case $x = k_3^n + nk_2h_2$, $y = k_2^n + nk_2h_2$, $z = ((k_1^n)/n) – nk_2h_2$

III. Conclusion

The positive numbers a,b and c satisfying $a^n+b^n=c^n$ will be of the form

1. $x = k_3^n + nk_1h_1$, $y = k_2^n + nk_1h_1$, $z = k_1^n - nk_1h_1$, if none of x,y,z is a multiple of n.
(This can also be written in the simple form
$x= (k_1^n - k_2^n + k_3^n)/2$, $y= (k_1^n +k_2^n - k_3^n)/2$, $z = (k_1^n +k_2^n + k_3^n)/2$)

2. $x = k_3^n + nk_2h_2$, $y = k_2^n + nk_2h_2$, $z = ((k_1^n)/n) – nk_2h_2$, if one and only one of x,y,z is a multiple of n

Acknowledgement

I am grateful to H.E. Dr.A.P.J.Abdul Kalam, Former President of India, Dr. N. Balakrishnan, Associate Director, IISc, Bangalore, Dr. R. Balasubrahmanian, Director, IMSc, Chennai, Dr. T. Thrivikraman Namboothiri, Former Head of the Department of Mathematics, Cochin University, Dr.V.K.Krishnan Namboothiri, Former Head of the Department of Mathematics, St. Thomas College, Thrissur and Prof. T.S.Balasubrahmanian, Former Head of the Department of Mathematics, Sreekrishna College, Guruvayur for their valuable comments and suggestions on my proof to find numbers a,b and c such that $a^n+b^n=c^n$.

References

[1]. Stewart, I. and Tall, D. Algebraic Number Theory . (Chapman & Hall, 1994)
[2]. www-gap.dcs.st-and.ac.uk/~history/HistTopics/Fermat's_last_theorem.html. Accessed on 02-08-2004. [online].