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Pythagorean Triangle with Area/ Perimeter as a special polygonal number

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Abstract: *Patterns of Pythagorean triangles, in each of which the ratio Area/ Perimeter is represented by some polygonal number. A few interesting relations among the sides are also given.*
Keyword: *Polygonal number, Pyramidal number, Centered polygonal number, Centered pyramidal number, Special number*

I. Introduction

The method of obtaining three non-zero integers x, y and z under certain conditions satisfying the relation $x^2 + y^2 = z^2$ has been a matter of interest to various Mathematicians [1, 3, 4, 5,6]. In [7-15], special Pythagorean problems are studied. In this communication, we present yet another interesting Pythagorean problem. That is, we search for patterns of Pythagorean triangles where, in each of which, the ratio Area/ Perimeter is represented by a special polygonal number. Also, a few relations among the sides are presented.

Notation

p_n^m = Pyramidal number of rank n with sides m

$t_{m,n}$ = Polygonal number of rank n with sides m

jal_n = Jacobsthal Lucas number

ja_n = Jacobsthal number

$ct_{m,n}$ = Centered Polygonal number of rank n with sides m

cp_n^m = Centered Pyramidal number of rank n with sides m

g_n = Gnomonic number of rank n with sides m

p_n = Pronic number

$carl_n$ = Carol number

ky_n =Kynea number

II. Method of Analysis

The most cited solution of the Pythagorean equation,

$$x^2 + y^2 = z^2 \qquad (1)$$

is represented by

$$x = 2uv,\ \ y = u^2 - v^2,\ \ z = u^2 + v^2 \left(u > v > 0\right) \qquad (2)$$

Pattern 1:

Denoting the Area and Perimeter of the triangle by A and P respectively, the assumption

$$\frac{A}{P} = t_{11,n}$$

leads to the equation

$$q\left(p - q\right) = n\left(9n - 7\right)$$

This equation is equivalent to the following two systems I and II respectively:

p-q	Q
9n-7	N
N	9n-7

In what follows, we obtain the values of the generators p, q and hence the corresponding sides of the Pythagorean triangle

Case 1:

On evaluation, the values of the generators satisfying system I are,

$$p = 10n - 7, q = n$$

Employing (2), the sides of the corresponding, Pythagorean triangle are given by

$$x(n) = 20n^2 - 14n,\ y(n) = 99n^2 - 140n + 49,\ z(n) = 101n^2 - 140n + 49$$

Examples:

n	X	Y	Z	A	P
1	6	8	10	24	24
2	52	165	173	4290	390
3	138	520	538	35880	1196
4	264	1073	1105	141636	2442
5	430	1824	1874	392160	4128

Properties:

1) $6\left\{\frac{(z-y)x - 8n^3}{t_{22,n}}\right\}$ is a Nasty number[2]

2) $10x - y - z \equiv 42 \pmod{140}$

3) $\left(10x(2n+1) - y(2n+1) - z(2n+1) + 78\right)^2 = 140^2\left(8t_{3,n+1}\right)$

Case 2:

On evaluation, the values, of the generators satisfying system II are

$$p = 10n - 7, q = 9n - 7$$

Using (2), the corresponding Pythagorean triangle is

$$x(n) = 180n^2 - 266n + 98,\ y(n) = 19n^2 - 14n,\ z(n) = 181n^2 - 266n + 98$$

Examples:

n	X	Y	z	A	P
1	12	5	13	30	30
2	286	48	290	6864	624
3	920	129	929	59340	1978
4	1914	248	1930	1081	2372
5	3268	405	3293	661770	6966

Properties:

1) $z - 9y - s_n - t_{10,n} \equiv 97 \pmod{131}$

2) $10y - z - x + 5t_{65,n} + p_n^5 \equiv 98 \pmod{228}$

3) $10x - 9(y+z) \equiv 98 \pmod{140}$

Pattern 2:

The assumption

$$\frac{A}{P} = t_{12,n}$$

leads to the equation

$$q(p-q) = 2n(5n-4)$$

This equation is equivalent to the following two systems I and II respectively

p-q	Q
5n-7	2n
2n	5n-7

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of the generators satisfying system I are,

$$p = 7n - 4, \ q = 2n$$

Using (2), the corresponding Pythagorean triangle is

$$x(n) = 28n^2 - 16n, \ y(n) = 45n^2 - 56n + 16, \ z(n) = 53n^2 - 56n + 16$$

Examples:

n	X	Y	z	A	P
1	12	5	13	30	30
2	80	84	116	3360	280
3	204	253	325	25806	782
4	384	512	640	98304	1536
5	620	861	1061	266910	2542

Properties:

1) $x(z - y) - 8t_{58,n} \equiv 0 \pmod{88}$

2) $x + y - z - 4t_{12,n} = 0$

3) $6\left\{\dfrac{y + 8n^2}{z}\right\}$ is a nasty number

Case 2:

On evaluation, the values of the generators satisfying the system II are

$$p = 7n - 4, \ q = 6n - 4$$

Using (2), the corresponding Pythagorean triangle is

$$x(n) = 84n^2 - 104n + 32, \ y(n) = 13n^2 - 8n, \ z(n) = 85n^2 - 104n + 32$$

Examples:

n	x	Y	z	A	P
1	12	5	13	30	30
2	160	44	164	3520	368
3	476	109	485	25942	1070
4	960	200	976	96000	2136
5	1612	317	1637	255502	3566

Properties:

1) $(z - x)y - 2ct_{13,n^2} + t_{44,n} \equiv -2 \pmod{20}$

2) $x - 6y + 32g_n - 8n$ is a Nasty number

3) $x(n+1) - 6y(n+1) - 12t_{3,n+1} \equiv 30 \pmod{62}$

Pattern 3:

Under our assumption

$$\frac{A}{P} = t_{13,n}$$

leads to the equation

$$q(p - q) = n(11n - 7)$$

This equation is equivalent to the following two systems I and II,

p-q	Q
11n-7	N
n	11n-7

Case 1:

On evaluation, the values of the generators satisfying system I are

$$p = 12n - 7,\ \ q = n$$

In view of (2) the corresponding Pythagorean triangle is

$$x(n) = 24n^2 - 14n\,,\ y(n) = 143n^2 - 168n + 49,\ z(n) = 145n^2 - 168n + 49$$

Examples:

n	x	Y	z	A	P
1	10	24	26	120	60
2	68	285	293	9690	646
3	174	832	850	72384	1856
4	328	1665	1697	273060	3690
5	530	2784	2834	737760	6148

Properties:

1) $6\left\{\dfrac{y + 2n^2}{z}\right\}$ is a Nasty number

2) $z - 6x - 2t_{3,n} \equiv 49 \pmod{85}$

3) $n(z - y) + x - 6p_n^4 - 2t_{23,n} \equiv 0 \pmod{4}$

Case 2:

On evaluation, the values of the generators satisfying system II are

$$p = 11n - 7,\ \ q = 10n - 7$$

Employing (2), the corresponding Pythagorean triangle is

$$x = 220n^2 - 294n + 98\,,\ y = 21n^2 - 14n\,,\ z = 221n^2 - 294n + 98$$

Examples:

n	x	Y	z	A	P
1	24	7	84	56	25
2	390	56	10920	840	394
3	1196	147	87906	2548	1205
4	2442	280	341880	5180	2458
5	4128	455	939120	8736	4153

Properties:

1) $x - 10y - t_{22,n} \equiv 98 \pmod{145}$

2) $n(z - 10y) - 6p_n^{13} + 52t_{8,n} + 2t_{3,n} \equiv 0 \pmod{3}$

3) $(z - x)(x - 13y) = 2t_{15,n^2} - 61cp_n^6 + 506p_n^5$

Pattern 4:

Under the assumption

$$\frac{A}{P} = t_{14,n}$$

leads to the equation

$$q(p - q) = 2n(6n - 5)$$

This equation is equivalent to the following two systems

p-q	Q
6n-7	2n
2n	6n-5

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 8n - 5, \quad q = 2n$$

Employing (2), the corresponding Pythagorean triangle is

$$x = 32n^2 - 20n, \; y = 60n^2 - 80n + 25, \; z = 68n^2 - 80n + 25$$

Examples:

n	x	Y	z	A	P
1	12	5	13	30	30
2	88	105	137	4620	330
3	228	325	397	37050	950
4	432	665	793	143640	1890
5	700	1125	1325	393750	3150

Properties:

1) $z - 2x - t_{9,n} \equiv 25 \pmod{37}$

2) $n(z - y) = 8cp_n^6$

3) $nx - 6cp_n^{30} - 2p_n^8 + t_{44,n} \equiv 0 \pmod{5}$

Case 2:

On evaluation, the values of generators satisfying II are

$$p = 8n - 5, \quad q = 6n - 5$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 96n^2 - 140n + 50, \; y(n) = 28n^2 - 20n, \; z(n) = 100n^2 - 140n + 50$$

Examples:

n	x	Y	z	A	P
1	6	8	10	24	24
2	154	72	170	5472	1216
3	494	192	530	47424	1216
4	1026	368	1090	188784	2484
5	1750	600	1850	525000	4200

Properties:

1) $\frac{x - 3y}{2} + 20g_n - 5$ is a Nasty number

2) $\frac{z}{2} - ct_{23,n} \equiv 23 \pmod{116}$

3) $\frac{x}{2} + 3y - 23 = 6ct_{8,n} - 2g_n$

Pattern 5:

The assumption

$$\frac{A}{P} = t_{15,n}$$

leads to the equation

$$q(p - q) = n(13n - 11)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
13n-11	N
n	13n-11

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 14n - 11, \quad q = n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 28n^2 - 22n, \quad y(n) = 195n^2 - 308n + 121, \quad z(n) = 197n^2 - 308n + 121$$

Examples:

n	x	Y	z	A	P
1	6	8	10	24	24
2	68	285	293	9690	646
3	186	952	970	88536	2108
4	360	2009	2041	361620	4410
5	590	3456	3506	1019520	7552

Properties:

1) $z - 7x - 42cp_n^{28} + 196cp_n^{6} - 121$ is a perfect square

2) $\left(\dfrac{z-y}{2}\right)^2$ is biquadratic integer

3) $\dfrac{nx}{2} - 6cp_n^{13} + 3t_{24,n} + 3n$ is cubic integer

Case 2:

On evaluation, the values of generators satisfying system II are

$$p = 14n - 11, \quad q = 13n - 11$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 364n^2 - 594n + 242, \quad y(n) = 27n^2 - 22n, \quad z(n) = 365n^2 - 594n + 242$$

Examples:

n	x	Y	z	A	P
1	12	5	13	30	30
2	510	64	514	16320	1088
3	1736	177	1745	153636	3658
4	3690	344	3706	634680	7740
5	6372	565	6397	1800090	13334

Properties:

1) $(z - x)(x - 13y) = 2t_{15,n^2} - 561cp_n^{6} + 506p_n^{5}$

2) $(z - x)y = n^2 t_{56,n} + 4cp_n^{6}$

3) $14y - z - 2t_{15,n} \equiv -242 \pmod{275}$

Pattern 6:

The assumption

$$\frac{A}{P} = t_{16,n}$$

leads to the equation

$$q(p - q) = 2n(7n - 6)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
7n-6	2n
2n	7n-6

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 9n - 6,\ \ q = 2n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 36n^2 - 24n\,,\ y(n) = 77n^2 - 108n + 36,\ z(n) = 85n^2 - 108n + 36$$

Examples:

n	x	y	z	A	P
1	12	5	13	30	30
2	96	128	160	6144	384
3	252	405	477	51030	1134
4	480	836	964	200640	2280
5	780	1421	1621	554190	3822

Properties:

1) $8x - 3y - 2t_{58,n} \equiv -108 \pmod{186}$
2) $x - t_{58,n} + 8n^2 \equiv 0 \pmod{3}$
3) $z - y - ct_{16,n} \equiv -1 \pmod{8}$

Case 2:

On evaluation the values of generators satisfying equation II are

$$p = 9n - 6,\ \ q = 7n - 6$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 126n^2 - 192n + 72\,,\ y(n) = 32n^2 - 24n\ ,\ z(n) = 130n^2 - 192n + 72$$

Examples:

n	x	y	z	A	P
1	6	8	10	24	24
2	192	80	208	7680	480
3	630	216	666	68040	1512
4	1320	416	1384	274560	3120
5	2262	680	2362	769080	5304

Properties:

1) $4y - 2x + z - 48g_n + 24n$ is a Nasty number
2) $y(2n+1) - x - ct_{4,n} \equiv -65 (\mathrm{mod}\, 294)$
3) $z(n-1) - x - ct_{8,n} + 132g_n = 189$

Pattern 7:

The assumption

$$\frac{A}{P} = t_{17,n}$$

leads to the equation

$$q(p - q) = n(15n - 13)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
15n-13	N
n	15n-13

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 16n - 13, \quad q = n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 32n^2 - 26n,\ y(n) = 255n^2 + 169 - 416n,\ z(n) = 257n^2 + 169 - 416n$$

Examples:

n	x	y	z	A	P
1	6	8	10	24	24
2	76	357	365	13566	798
3	210	1216	1234	127680	2660
4	408	2585	2617	527340	5610
5	670	4464	4514	1495440	9648

Properties:

1) $x(2^n) - 8jal_{2n} + 26mer_n + 34 = 0$

2) $(y - z)\left(2^n\right) - 2 = carl_n + ky_n$

Case 2:

On evaluation the values of generators satisfying system II are given by

$$p = 16n - 13, \quad q = 15n - 13$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 480n^2 - 806n + 338,\ y(n) = 31n^2 - 26n,\ z(n) = 481n^2 - 806n + 338$$

Examples:

n	x	y	z	A	P
1	12	5	13	30	30
2	358	72	650	12888	1080
3	2240	201	2249	225120	4690
4	4794	392	4810	939624	9996
5	8308	645	8333	2679330	17286

Properties:

1) $y - 31p_n^5 \equiv 0 \pmod{57}$

2) $y - z + x - t_{62,n} \equiv 0 \pmod{3}$

3) $(x - 15y)(z - x) - 4\left(ct_{3,n}.ct_{5,n}\right) + 223nt_{6,n} - 7t_{26,n} \equiv -4 \pmod{61}$

Pattern 8:

Under our assumption

$$\frac{A}{P} = t_{18,n}$$

leads to the equation

$$q(p - q) = 2n(8n - 7)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
8n-7	2n
2n	8n-7

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 10n - 7, \quad q = 2n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 40n^2 - 28n,\ y(n) = 96n^2 - 140n + 49,\ z(n) = 104n^2 - 140n + 49$$

Examples:

n	x	y	z	A	P
1	12	5	13	30	30
2	104	153	185	7956	442
3	276	493	565	68034	1334
4	528	1025	1153	270600	2706
5	860	1749	1949	752070	4558

Properties:

1) $x - t_{60,n} + 11n^2 = 0$

2) $n(y - 2x) - Rh_n + t_{46,n} + 26n^2 = 7$

3) $(z - y)x - z + 18t_{26,n} + 141g_n + 190 = 0$

Case 2:

On evaluation, the values of generators satisfying system I are

$$p = 10n - 7, \quad q = 2n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n) = 160n^2 - 252n + 98,\ y(n) = 36n^2 - 28n,\ z(n) = 164n^2 - 252n + 98$$

Examples:

n	x	y	z	A	P
1	6	8	10	24	24
2	234	88	250	10296	572
3	782	240	818	93840	1840
4	1650	464	1714	382800	3828
5	2838	760	2938	1078440	6536

Properties:

1) $\dfrac{(z - x)(2n + 1)}{2} = ct_{23,n} + t_{11,n} + 1$

2) $5x - 2y - z = 76g_n - 4t_{50,n} - 169$

3) $(x - 4y)(n - 1) - 2(t_{17,n} + t_{3,n}) + 80g_n + 174 = 0$

Pattern 9:

Under our assumption

$$\frac{A}{P} = t_{19,n}$$

leads to the equation

$$q(p - q) = n(19n - 17)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
19n-17	N
n	19n-17

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p = 20n - 17, \quad q = n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n)=40n^2-34n\,,\ y(n)=399n^2-680n+289,\ z(n)=410n^2-680n+289$$

Examples:

n	x	y	z	A	P
1	6	8	10	24	24
2	92	525	533	24150	1150
3	258	1840	1858	237360	3956
4	504	3953	3958	996156	8442
5	830	6864	6914	2848560	14608

Properties:

1) $(z-y)\left(2^n\right)=jal_{2n}+3ja_{2n}$

2) $x-2t_{42,n}\equiv 0(\bmod 4)$

3) $(z-y)(2n+1)-2\left(t_{3,n+1}+ct_{3,n}-2\right)+g_n=0$

Case 2:

On evaluation, the values of generators satisfying system I are

$$p=20n-17,\quad q=19n-17$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n)=760n^2-1326n+578\,,\ y(n)=39n^2-34n,\ z(n)=761n^2-1326n+578$$

Examples:

n	x	y	z	A	P
1	12	5	13	30	30
2	966	88	970	42504	2024
3	3440	249	3449	428280	7138
4	7434	488	7450	1813896	15372
5	12948	805	12973	5211570	26726

Properties:

1) $y\left(2^n\right)=28carl_n+11ky_n+39$

2) $20y-z-2t_{21,n}+274g_n+852=0$

3) $39(z-x)-y\equiv 0(\bmod 34)$

Pattern 10:

Under our assumption

$$\frac{A}{P}=t_{20,n}$$

leads to the equation

$$q(p-q)=2n(5n-4)$$

The above equation is equivalent to the following two systems I and II respectively

p-q	Q
5n-4	2n
2n	5n-4

As in the previous case, we obtain the values of the generators u, v and hence the corresponding sides of the Pythagorean triangle.

Case 1:

On evaluation, the values of generators satisfying system I are

$$p=7n-4,\quad q=2n$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n)=28n^2-16n\,,\ y(n)=45n^2-56n+16,\ z(n)=53n^2-56n+16$$

Examples:

n	x	y	z	A	P
1	12	5	13	30	30
2	80	84	116	3360	280
3	204	253	325	25806	782
4	384	512	640	98304	1500
5	620	861	1061	266910	2542

Properties:

1) $(z-y)x+245cp_n^6=t_{66,n^2}+42p_n^5$

2) $3y-2x-z-t_{54,n}\equiv 32\pmod{55}$

3) $\dfrac{((2x-z)+y)(n-1)+49g_n-28}{t_{3,n+1}+t_{17,n}}$ is a nasty number

Case 2:

On evaluation, the values of generators satisfying system II are

$$p=7n-4,\quad q=5n-4$$

Employing (2), the corresponding Pythagorean triangle is

$$x(n)=70n^2-96n+32,\ y(n)=24n^2-16n,\ z(n)=74n^2-96n+32$$

Examples:

n	x	y	z	A	P
1	6	8	10	24	24
2	120	64	136	3840	320
3	374	168	410	31416	952
4	768	320	832	122880	1920
5	1302	520	1402	338520	3224

Properties:

1) $x+z-6y+48g_n=16$

2) $z-x=2t_{3,2n+1}$

3) $z-3y-ct_{4,n}\equiv 31\pmod{50}$

III. Conclusion

One may search for other patterns of Pythagorean triangles under consideration.

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