Cash Flow Valuation Model in Continuous Time

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Abstract: This paper presents valuation models which are defined as the expected discounted value of a stream of each flow at a time. Three equivalent forms of this value process are established with each of which has its own merits. Local dynamics of the value process is also considered.

Keywords: Cash Flow, Valuation, Continuous, Deflator, Filtration.

I. Introduction

The advantage of the stochastic calculus of semimartingales, especially on Ito diffusion model gives us an opportunity to consider the cash flow in continuous time. When an individual or a firm is faced with a stream of cash flows \((C_t)_{t \geq 0}\), we define the value of such a stream at time \(t\) as

\[
V_t = E \left[ \int_t^\infty C_s e^{-r(s-t)} ds \right],
\]

where \(E_t[\cdot]\) denotes the expectation with respect to all known information up to time \(t\), and \(r\) is some constant discount rate. We can re-write the above \(V_t\) as follows:

\[
\begin{align*}
V_t &= E_t \left[ \int_t^\infty C_s e^{-r(s-t)} ds \right] \\
V_t &= E_t \left[ e^{r \int_0^t C_s ds} \right] \\
V_t e^{-rt} &= E_t \left[ \int_0^t C_s e^{-r(s-t)} ds \right] \\
V_t e^{-rt} &= E_t \left[ \int_0^t C_s e^{-r(s-t)} ds - \int_0^t C_s e^{-rs} ds \right]
\end{align*}
\]

Thus,

\[
V_t e^{-rt} = E_t \left[ \int_0^t C_s e^{-rs} ds \right] - \int_0^t C_s e^{-rs} ds
\]

Assuming \(E[\int_0^\infty C_s e^{-rs} ds] < \infty\), then this is a decomposition where the discounted present value is the sum of a uniformly integrable martingale and a predictable process. If we let \(M\) denote the martingale then (2) becomes

\[
d(V_t e^{-rt}) = dM_t - C_t e^{-rt} dt
\]

Using differential rule for products, the dynamic of the present value \(V_t\) is given as

\[
e^{-rt} dV_t + V_t e^{-rt} dt = dM_t - C_t e^{-rt} dt
\]

\[
e^{-rt} dV_t - rV_t e^{-rt} dt = dM_t - C_t e^{-rt} dt
\]

\[
\Rightarrow dM_t - rV_t e^{-rt} dt = e^{-rt} dM_t - C_t dt
\]

Let \(A_t = e^{-rt}\) then (1) becomes

\[
\begin{align*}
V_t &= E_t \left[ \int_t^\infty C_s e^{-rs} ds \right] \\
V_t A_t &= E_t \left[ \int_t^\infty C_s A_s ds \right]
\end{align*}
\]

Thus, for \(h > 0\), we have

\[
V_t A_t = E_t \left[ \int_t^{t+h} C_s A_s ds + \int_{t+h}^\infty C_s A_s ds \right]
\]

\[
= V_t A_t = E_t \left[ \int_t^{t+h} C_s A_s ds + \int_{t+h}^{t+2h} C_s A_s ds \right]
\]

\[
0 = E_t \left[ \int_t^{t+h} C_s A_s ds + V_{t+h} A_{t+h} - V_t A_t \right]
\]

\[
0 = E_t \left[ \int_t^{t+h} C_s A_s ds + E_t[V_{t+h} A_{t+h} - V_t A_t] \right]
\]

let \(h \to 0\) then

\[
0 = C_t A_t dt + E_t[d(V_t A_t)]
\]

The introduction of \(A_t\) is to allow for more general discount factors, especially stochastic ones. Also, we want to generalize the cash flows, allowing process of finite variation as integrators with which we integrate deflator.

1.1 Preamble

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a complete probability space equipped with a filtration \((\mathcal{F}_t)_{t \geq 0}\). The filtration is assumed to be right continuous and complete. Any adapted process will be adapted with respect to the filtration \(\mathcal{F}_t\). Let
\( \mathcal{F}_{\omega} \) be \( \sigma \)-algebra \( V_{\omega} \geq 0 \mathcal{F} \), and assume that \( \mathcal{F}_{\omega} = \mathcal{F} \). Thus a process is cadlag, if almost every samples path of the process is right continuous with left limits.

A process where path are a.s positive increasing and right continuous is said to be an increasing process. An increasing function has left limits and thus any increasing process is endlag.

For every increasing process \( A \) we use \( A_{\omega} = 0 \) a.s but not necessary that \( A_{\omega} = 0 \) a.s. A process is of finite variation if it is cadlag and adapted and if almost every sample path is of finite variation on each compact subset of \([0, \infty)\).

A process is of finite variation if it is the difference between two increasing processes. A process \( X \) is said to be optional if the mapping \( X : [0, \infty) \rightarrow \mathcal{R} \) is measurable whenever \([0, \infty) \times \Omega\), is given as the optional \( \sigma \)-algebra.

Since the optional \( \sigma \)-algebra is generated by the family of all adapted process which are cadlag. Thus, every finite variation process \( (FV) \), \( A \) is optional. Assuming \( A \) is a finite variation process and \( X \) is a real valued process on \([0, \infty) \times \Omega \) that is \( B \times \mathcal{F} \) - measurable (where \( B \) denotes the Borel \( \sigma \)-algebra on \([0, \infty)\) then stieltjes integral is defined as

\[
(X \cdot A)_{t}(\omega) = \int_{[t_{0}, t]} X_{s}^{\omega}(\omega) dA_{s}^{\omega}(\omega)
\]

whenever it exists. If \( X \cdot A \) exists for all \( t \in [0, \infty) \) and almost all \( \omega \in \mathcal{Q} \) then \( (X, A)_{t} \), defines a process with \( (X, A)_{0} = X_{0}A_{0} \). If \( X \) is optional, then there is an optional version of \( (X, A) \). If \( A \) is an FV process, then \( |A|_{t} = \int_{0}^{t} |dA_{s}(\omega)| \) denotes the total variation of \( A \). \( |A| \) is adapted and cadlag and \( |A|_{0} = |A|_{\infty} \). If \( A \) is an FV process and \( X \) is measurable process then the integral \( \int_{0}^{t} X_{s}^{}(\omega) d |A|_{s}(\omega) \) denotes integration with respect to \( d|A| \). A semimartingale is an adapted and cadlag process \( (X_{\omega})_{t \geq 0} \) having a decomposition \( X_{t} = X_{0} + M_{t} + A_{t} \) where \( M \) is a local martingale and \( A \) is an FV process.

II. Cash flows and deflators

Definition 1: A cash flow process is a \( C_{\omega}^{t}(\omega) \) is a finite variation process.

This definition of the cash flow process makes it trivially a semimartingale. It can also be used as integrator.

Definition 2: A deflator is a strictly positive semimartingale that is finite a.s.

Deflator is considered to be a semimartingale in order to make differentiation rule valid for semimartingales. It is noted that if \( A \) is a deflator then both \( 1/A \) and \( \ln A \) are well defined, and since \( 1/x \) and \( \ln x \) are twice continuously differentiable on \((0, \infty)\), both \( 1/A \) and \( \ln A \) are semiartingales.

Definition 3: Given a deflator \( A \), the discount process implied by \( A \) is defined by

\[
m(s, t) = \frac{A(t)}{A(s)} \quad s, t \geq 0
\]

The proposition below presents some important properties of the discount process. It is easier to work with deflator as defined above in continuous time.

Proposition 1: Let \( m \) be a discount process implied by the deflator \( A \). Then

i. \( m(s, t, \omega) \) is \( \mathcal{F}_{\omega}^{\max(t, s)} \) measurable for every \( s, t \in [0, \infty) \)
ii. \( 0 < m(s, t) < \infty \) a.s for every \( s, t \in [0, \infty) \)
iii. \( m(s, t) = m(s, u) m(u, t) \) a.s for every \( 0 \leq s \leq u \leq t \)

A discount process fulfilling \( 0 < m(s, t) < 1 \) a.s. for every \( s, t \in [0, \infty) \) will be termed as normal discount process. Hence, \( m \) is normal discount process. Thus, \( m \) is normal if and only if \( A \) is non-decreasing. Thus, a discount process \( m \) with deflator \( A \) can be written in the form

\[
m(s, t, \omega) = \exp (-\int_{t}^{s} \lambda(\omega) du)
\]

If and only if \( \ln A(t, \omega) \) is absolutely continuous in \( t \) for almost every \( \omega \in \Omega \) with density \(-A(t, \omega)\).

III. Valuation

Definition 4: Given a cash flow process \( C \) and a deflator \( A \) such that

\( E[\int_{[t, \infty)} A_{\omega} C_{\omega} d\omega] \) is finite the value process is defined for \( t \in [0, \infty) \) as

\[
V_{t} = \frac{1}{A_{t}} E[\int_{[0, t]} A_{\omega} C_{\omega} d\omega | \mathcal{F}_{t}]
\]

By noting that \( \int_{[0, t]} = \int_{[0, \infty]} + \int_{[t, \infty]} \) and using the fact that every optimal we have

\[
V_{t}A_{t} = E[\int_{[t, \infty]} A_{\omega} C_{\omega} d\omega | \mathcal{F}_{t}]
\]
\[ V_t A_t = E\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] = E\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] - \int_{[0,\infty)} A_s dC_s \]
\[ = M_t - (A \cdot C)_t \]

Since \( E \mid M_t = I \mid E \left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \leq E\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] < \infty \)

for every \( t \in [0, \infty) \), \( M_t \) is a uniformly integrable martingale. The filtration \( (\mathcal{F}_t) \) is right continuous, thus there exists a modification of \( M \) that is right continuous. Recall, \( \lim_{t \to \infty} M_t = \int_0^\infty A dC_s \) a.s. Since \( M \) and \( A \cdot C \) are right continuous and adapted the value process is also continuous and adapted. Thus, the value process is optional. From (3) we have

\[ V_t = \frac{M_t}{A_t} - (A \cdot C)_t \]

Since \( M \) is a (true) Martingale, it is especially a semimartingale. \( A \cdot C \) is a process of finite variation and is thus also a semimartingale. Since \( A \) is a strictly positive semimartingale hence \( 1/A \) is also a strictly positive semimartingale and since the product of two semimartingales is a semimartingale. Hence, \( V \) is a semimartingale.

As in discrete time case there exists three equivalent ways of writing value process. Assumption made here are measurability condition on \( C \) and \( A \) and integrability condition making \( M \) into a uniformly integrable martingale. In addition to this, there must be a condition stating that the discounted value goes to zero as \( t \) tends to infinity (\( \infty \)) i.e \( V_t A_t \to 0 \) as \( t \to \infty \) a.s.

**Theorem 2:** Let \( C \) and \( A \) be a cash flow process and deflator respectively such that

\[ E\left[ \int_{[0,\infty)} A_t dC_s \mid \mathcal{F}_t \right] < \infty \]

Then the following three statement are equivalent

(i) for every \( t \in [0, \infty) \)
\[ V_t = \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]  \hspace{1cm} (4)

(ii) for every \( t \in [0, \infty) \)
\[ M_t = V_t A_t + \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]  \hspace{1cm} (5)

is a uniformly integrable martingale and

(b) \( V_t A_t \to 0 \) a.s when \( t \to \infty \)

(iii) for each \( t \in [0, \infty) \) we have

(a) for every \( h > 0 \)
\[ V_t A_t = E\left[ V_{t+h} A_{t+h} + \int_{(t,t+h]} A_s dC_s \mid \mathcal{F}_t \right] \]  \hspace{1cm} and

(b) \( \lim_{T \to \infty} E\left[ V_{t+T} A_{t+T} \mid \mathcal{F}_t \right] = 0 \)

**Proof:**

We begin by showing that (i) \( \iff \) (ii) and (i) \( \iff \) (iii)

(i) \( \iff \) (ii) To prove the if part

\[ \text{Using the fact that} \]
\[ \int_{[0,\infty]} = \int_{[0,\infty]} + \int_{[0,\infty]} \]

Thus,

\[ V_t = \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]

\[ V_t = \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] - \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]

\[ V_t = \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] - \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]

Note that \( \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] = M_t \)

it implies that

\[ V_t = \mathbb{E}\left[ \int_{[0,\infty)} A_s dC_s \mid \mathcal{F}_t \right] \]

Thus,

\[ M_t = \int_0^\infty A_t dC_s \text{ a.s as } t \to \infty \]

Thus,
Turning to the only if ‘part’. That is (ii) ⇒ (i). Let $t \to \infty$ in equation (5)

As $t \to \infty$, $M_\omega \to M_\omega$, $V_\omega \to 0$

Thus, $M_\omega = \int_{[0,\omega]} A_s dC_s$

Taking conditional expectation with respect to the $\sigma-$ algebra $\mathcal{F}_t$

i.e.

$E[M_\omega | \mathcal{F}_t] = E[\int_{[0,\omega]} A_s dC_s | \mathcal{F}_t]$

Thus,

$V_\omega A_t = E[\int_{[\tau,\omega]} A_s dC_s | \mathcal{F}_t]$

which is our desire result.

(i) ⇔ (ii). For ‘if’ part take $h > 0$. From equation (4)

$V_\omega A_t = E[\int_{[\tau,\omega]} A_s dC_s | \mathcal{F}_t]$

and $\int_{[0,\omega]} A_s dC_s$ is integrable, thus, using the dominated convergence theorem to get for every $A \in \mathcal{F}_T$

$\lim_{T \to \infty} E[V_\omega A_t + \int_{[0,\omega]} A_s dC_s | \mathcal{F}_T] = 0$

where the last equality follows from the fact that $\int_{[0,\omega]} A_s dC_s$ is finite a.s.

To prove the other direction of the equivalence for $T > 0$ from

$V_\omega A_t = E[\int_{[\tau,\omega]} A_s dC_s | \mathcal{F}_T] = E[\int_{[\tau,\omega]} A_s dC_s | \mathcal{F}_T]$

and $\int_{[0,\omega]} A_s dC_s$ is integrable, thus, using the dominated convergence theorem to get for every $A \in \mathcal{F}_T$

$\lim_{T \to \infty} E[V_\omega A_t + \int_{[0,\omega]} A_s dC_s | \mathcal{F}_T] = 0$

Using the theorem of dominated convergence we have

$V_\omega A_t = \lim_{T \to \infty} E[\int_{[\tau,\omega]} A_s dC_s | \mathcal{F}_T]$

IV. Local dynamics of the value process

To begin on the local dynamics of the value process, we start from the relation

$V_\omega A_t = M_\omega - A_\omega C_t$

Since all the process in the above expression are semimartingales thus differentiation rule for product of semimartingale can therefore be used.

Hence, we have from

$d(A_\omega C_t) = V_\omega dA_t + A_\omega dV_t + d[A,V]_t$

Note that

$d(A_\omega C_t) = dM_\omega - A_\omega dC_t$

$\Rightarrow$

$d(M_\omega + A_\omega dC_t) = V_\omega dA_t + A_\omega dV_t + d[A,V]_t$

Note also that,

$\Delta A_\omega \Delta C_t = A_\omega \Delta C_t - A_\omega \Delta C_t$

$A_\omega \Delta C_t = \Delta A_\omega \Delta C_t + A_\omega \Delta C_t - A_\omega \Delta C_t$

Divide equation (6) by $A_\omega$ we get

$\frac{1}{A_\omega} d[V_\omega A_t] = V_\omega \frac{dA_t}{A_\omega} + dV_t + \frac{d[V,A]_t}{A_\omega}$

$\Rightarrow$

$dV_t = V_\omega \left(- \frac{dA_t}{A_\omega} \right) + \frac{1}{A_\omega} d[V_\omega A_t] - \frac{1}{A_\omega} d[V,A]_t$

Substitute for (7) and (9) in (10)
\[ dV_t = V_t^{- \left( -\frac{dAt}{At} \right)} + \frac{1}{\nu} (dMt - At dC_t) - \frac{1}{\nu} d(V, \Lambda)_t \]
\[ dV_t = V_t^{- \left( -\frac{dAt}{At} \right)} + \frac{1}{\nu} (dMt - \Delta \Lambda_t \Delta C_t - At dC_t) - \frac{1}{\nu} d(V, \Lambda)_t \]
\[ dV_t = V_t^{- \left( -\frac{dAt}{At} \right)} - dC_t + \frac{1}{\nu} (dMt - \Delta \Lambda_t \Delta C_t) - \frac{1}{\nu} d(V, \Lambda)_t \]

Let \( dR_t = -\frac{dAt}{At} \) and \( N_t = \int_0^t \frac{1}{\nu} dM_s \)

\[ dV_t = V_t^{-dR_t - dC_t + \frac{1}{\nu} \Delta \Lambda_t \Delta C_t + d(V, \Lambda)_t} \]

\( \Lambda \) becomes the stochastic exponential of \( R \) if \( \Lambda_0 = 1 \). Hence \( R \) can be referred to as discount rate associated with \( \Lambda \). For an improvement of economical interpretation of equation (6) note that if given a cash flow \( \tau^f_t dt \) for \( t \in [0, \infty) \), where \( \tau^f_t \) is measurable and adapted and such that for almost everywhere \( \omega \in \Omega \) we have \( 0 \leq \tau^f_t(\omega) \) for every \( t \in [0, \infty) \) and if the value process of this cash flow stream fulfills \( V^f_t \equiv 1 \). Then \( \tau^f_t \) can be referred to as a locally risk free interest rate if all these specification are inserted into equation (7)

We have from

\[ d(V_t, \Lambda_t) = dMt - At dC_t \]

\[ \tau^f_t dt = -\frac{dAt}{At} + \frac{1}{\Lambda_t} dMt \]

Where \( M^* \) is a martingale. Thus, if the deflator \( \Lambda \) assigns the cash flow stream given by \( \tau^f_t dt \) the value \( 1 \) for all \( t \in [0, \infty) \), then in using the same \( \Lambda \) for valuing another cash flow stream \( C \), the differential of \( V \) can be expressed in terms of the risk free rate.

From \( M_t = E[J_0^t \Lambda_s dC_s \mid F_t] \) inserting \( \tau^f_t dt \) we have

\[ M^* = E[J_0^t \Lambda_s \tau^f_t dC_s \mid F_t] \]

With \( V_t \equiv 1 \) and inserting \( \tau^f_t \) in equation (7) and integrate we have

\[ \Lambda_t = E[J_0^\infty \Lambda_t \tau^f_s dS] - \int_0^T \Lambda_t \tau^f_t dS \]

\[ \Lambda_t = M^* = \int_0^T \Lambda_t \tau^f_t dS \]

Since \( \tau^f_t \geq 0 \) a.s. thus \( \Lambda_t = \int_0^T \Lambda_t \tau^f_t dS \) is an increasing process. That means \( \Lambda \) is a non-negative supermartingale. An adapted and cadlag process is said to be potential if it is non-negative super martingale that tends to \( 0 \) a. s. as \( t \rightarrow \infty \).

Since

\[ E[\Lambda_t] = E[J_0^\infty \Lambda_s \tau^f_s dC_s \mid F_t] - \int_0^T E[\Lambda_s \tau^f_t] dS \rightarrow 0 \text{ a.s.} \]

As \( t \rightarrow \infty \) then the proposition below is proved.

**Proposition 3:** If there exists a risk-less asset, then the deflator pricing this asset is a potential.

Equation (11) can be termed equation for deflator \( \Lambda \). If the filtration \( F_t \) is generated by a Brownian motion, then every square integrable martingale can be written as Ito integral. Here, one more asset priced by \( \Lambda \) is needed to determine \( \Lambda \).

To express the differential of \( V \) in terms of the rate \( \tau^f_t \). Assuming \( \Lambda \) is a continuous process then

\[ dV_t - d\Lambda_t + dV_t + d(V, \Lambda)_t = dM_t - At dC_t \]

\[ dV_t = V_t^{- \left( -\frac{dAt}{At} \right)} - dC_t - \frac{1}{\Lambda_t} d(V, \Lambda)_t + \frac{1}{\Lambda_t} dM_t \]

Substitute for \( d\Lambda_t = dM_t - \tau^f_t dt \)

\[ dV_t = \frac{\nu}{\Lambda_t} dM_t + V_t \tau^f_t dt - dC_t - \frac{1}{\nu} d(V, \Lambda)_t + \frac{1}{\nu} dM_t \]

\[ dV_t + dC_t = V_t \tau^f_t dt - \frac{1}{\nu} d(V, \Lambda)_t + \frac{1}{\nu} dM_t + \frac{1}{\nu} dM^*_t \]

Thus,

\[ \frac{dV_t + dC_t}{V_t} = \tau^f_t dt - \frac{1}{\nu} d(V, \Lambda)_t + \frac{1}{\nu} dM_t - \frac{1}{\nu} dM^*_t \]

Taking conditional expectations

\[ E\left[ \tau^f_t dt \mid F_t \right] = \tau^f_t dt - E\left[ \frac{d(V, \Lambda)_t}{V_t} \mid F_t \right] \]

Assuming \( V \) is strictly positive and write \( d(V, \Lambda)_t = dV_t d\Lambda_t \)

Then

\[ E\left[ \frac{dV_t + dC_t}{V_t} \mid F_t \right] = \tau^f_t dt - E\left[ \frac{dV_t d\Lambda_t}{V_t} \mid F_t \right] \]
The left hand side is known as the instantaneous net return of the value process at time \( t \). The expected return of the value process has now been decomposed into risk-free part \( \rho t \) and risk premium part \( \mathbb{E} \left[ \frac{dR_t}{R_t} \right] \).

If \( \frac{dR_t}{V_t} \) and \( \frac{dL_t}{A_t} \) are negatively correlated, then there is a positive risk premium and if they are positively correlated then the risk premium is negative. Since the of a risky investment giving us the cash flow \( C \) is expected to have a return strictly greater than the risk-free rate, it is seen that \( \frac{dR_t}{V_t} \) and \( \frac{dL_t}{A_t} \) is expected to be positively correlated. The intuition is that a risky investment is desirable if its value is high in ‘bad’ states of the world (when we really need money) and low in ‘good’ states of the economy (where everything else is good).

An investment with such properties will have a high price (since demand for this desirable investment opportunity is high) and thus, a low expected return. Hence, \( \frac{dL_t}{A_t} \) is interpreted as a measure of how ‘bad’ a state of economy is.

To solve equation (11) when the risk – less rate is equal to the constant \( \gamma > 0 \). Inserting this in equation (4) of theorem 2.

We have for \( V_t \leq 1 \)

\[
A_t = E \left[ \int_0^T \gamma L_s \, ds \mid \mathcal{F}_t \right]
\]

(12)

This is regard as stochastic differential equation for discount factor \( A_t \), substituting for \( A_t = e^{-\gamma t} \) on the RHS

\[
E \left[ \int_0^T \gamma e^{-\gamma t} \, ds \mid \mathcal{F}_t \right] = [e^{-\gamma t}]^T = e^{-\gamma T} = A_t
\]

Hence \( A_t \) is also equivalent to equation (12).

Assuming \( K \) is \( IR \), then also \( Ke^{-\gamma t} \) is a solution. The proposition below shows that there exist other solution to equation (12)

**Proposition 3:** Let \( \gamma > 0 \) be a given real number and consider

\[
X_t = E \left[ \int_0^T \gamma X_s \, ds \mid \mathcal{F}_t \right]
\]

(13)

Then \( X_t \) is a solution to (13) if and only if it can be written as

\[
X_t = e^{-\gamma T} \left( \frac{x_0}{U_t} \right) + \int_0^T e^{-\gamma t} \, dM_t \mid \mathcal{F}_t
\]

(14)

for some uniformly integrable martingale \( M_t \).

The following Lemma is needed for the prove

**Lemma 4:** Let \( N \) be a local martingale that is cadlag and fulfills \( N_0 = 0 \) a.s and let \( U \) be a predictable increasingly process with \( U_0 > 0 \) a.s if

\[
\int_0^T \left( \frac{1}{U_t} \right) dN_t \text{ converges a.s as } t \to \infty \text{ and the limit is finite, then } \frac{N_t}{U_t} \to \infty \text{ a.s as } t \to \infty \text{ on } \{ U_\infty = \infty \}
\]

Now back to the proof of the above proposition.

**Proof:** consider a discount factor \( A \) of the form given in equation (14) and define the local martingale as

\[
N_t = \int_0^T e^{-\gamma t} \, dM_t \quad \text{with } N \text{ and } U_t = e^{-\gamma t} \text{ we have}
\]

\[
dN_t = e^{-\gamma t} \, dM_t
\]

\[
dM_t = \frac{1}{U_t^2} \, dN_t
\]

\[
M_t = \int_0^T \left( \frac{1}{U_t^2} \right) dN_t
\]

As \( t \to \infty \), \( \int_0^T \left( \frac{1}{U_t^2} \right) dN_t \to M_t \to M \epsilon \mathcal{L}
\]

Since \( P(U_\infty = \infty) = 1 \) the Lemma above can be concluded that

\[
\frac{1}{e^{-\gamma t}} N_t \to \infty \text{ a.s from } \frac{N_t}{U_t} \to 0
\]

With the use of differentiation rule for products

\[
d(e^{-\gamma t} (A_t + N_t)) = -\gamma e^{-\gamma t} (A_t + N_t) \, dt + dM_t
\]

Using integration by parts

\[
\int_0^T \gamma e^{-\gamma t} (A_t + N_t) \, ds = [-e^{-\gamma t} (A_t + N_t)]_T^0 + [M_t]_T^0
\]

\[
= M_0 - M_T - e^{-\gamma T} (A_T + N_T) + e^{-\gamma T} (A_0 + N_0)
\]

as \( T \to \infty \), \( e^{-\gamma T} (A_T + N_T) \to 0 \) a.s Then we have

\[
\int_0^T \gamma e^{-\gamma t} (A_t + N_t) \, ds = M_0 - M_T + e^{-\gamma T} (N_0 + N_T)
\]

\[
\Rightarrow e^{-\gamma T} (A_T + N_T) = \int_0^T \gamma e^{-\gamma t} (A_t + N_t) \, ds - M_0 + M_T
\]
Taking conditional expectation with respect to $\mathcal{F}_t$
\[ E[e^{-r_T f}(A_0 + N_T) \mid \mathcal{F}_t] = E\left[ \int_t^T \eta e^{-r_s} (A_s + N_s) \, ds \mid \mathcal{F}_t \right] - E[M_T \mid \mathcal{F}_t] + E[M_t \mid \mathcal{F}_t] \]
\[ e^{-r_T f}(A_0 + N_T) = E\left[ \int_t^T \eta e^{-r_s} (A_s + N_s) \, ds \mid \mathcal{F}_t \right] - M_T + M_t \]
\[ e^{-r_T f}(A_0 + N_T) = E\left[ \int_t^T \eta e^{-r_s} (A_s + N_s) \, ds \mid \mathcal{F}_t \right] \]
\[ \mathbf{O} \]
\[ \mathbf{O} \]

\[ A_t = E\left[ \int_0^T \eta (A_s \mid \mathcal{F}_t) \, ds \mid \mathcal{F}_t \right] \]

By letting $t = 0$
\[ E\left[ \int_t^T \eta e^{-r_s} (A_s + N_s) \, ds \mid \mathcal{F}_t \right] \leq |A_0| + E[|M_T|] + |M_t| < \infty \]

In conclusion, $e^{-r_T f}(A_0 + N_T)$ is a solution.

Assuming $A_t$ is a solution
\[ A_t = E\left[ \int_t^T \eta (A_s \mid \mathcal{F}_t) \, ds \mid \mathcal{F}_t \right] \]
\[ A_t = E\left[ \int_t^T \eta (A_s) \, ds \mid \mathcal{F}_t \right] + \int_0^T \eta (A_s) \, ds \mid \mathcal{F}_t \]
\[ A_t = E\left[ \int_t^T \eta (A_s) \, ds \mid \mathcal{F}_t \right] - \int_0^T \eta (A_s) \, ds \]

Introducing U. I martingale $M_t = E\left[ \int_0^T \eta (A_s) \, ds \mid \mathcal{F}_t \right]$ we have
\[ A_t e^{-r_T f} = A_0 + \int_0^T \eta \Delta_t e^{-r_T f} \, ds \]
which is our desired result.

V. Conclusion:

We have been able to write out the three equivalent ways of cash flow each of which has its own merit. While considering the local dynamics of the value process it’s discovered that to have a locally risk free interest rate the value process of the cash flow stream must fulfill $V_t = 1$

REFERENCES