IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728,p-ISSN: 2319-765X, Volume 6, Issue 4 (May. - Jun. 2013), PP 35-42
www.iosrjournals.org

Cordial Labelling Of K-Regular Bipartite Graphs for K = 1, 2, N, N-1 Where K Is Cardinality of Each Bipartition

Pranali Sapre
JJT University, Jhunjhunu, Rajasthan, India
Assistant Professor,Vidyavardhini's College of Engineering & Technology
Vasai Road Dist. Thane. Maharashtra

***Abstract**: In the labelling of graphs one of the types is cordial labelling. In this we label the vertices 0 or 1 and then every edge will have a label 0 or 1 if the end vertices of the edge have same or different labellings respectively. Here we are going find whether a k-regular bipartite graph can be cordial for different values of k.*

I. Introduction :

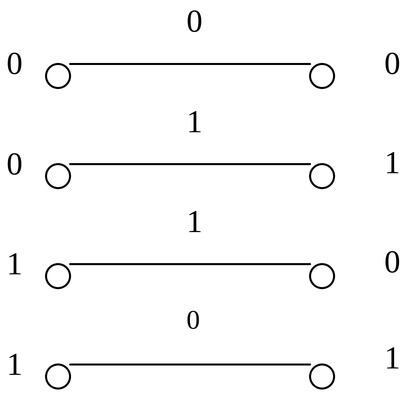
- Regular graph: A graph G is said to be a regular graph if degree of each vertex a same. It is called k-regular if degree of each vertex is k.
- Bipartite graph: Let G be a graph. If the vertex of G is divided into two subsets A and B such that there is no edge 'ab' with $a,b \in A \text{ or } a,b \in B$ then G is said to be bipartite that is, Every edge of G joins a vertex in A to a vertex in B.
  The sets A and B are called partite sets of G.
- Complete graph: A graph in which every vertex is adjacent to every other vertex is a complete graph. For a complete graph on n vertices degree of each vertex is n-1.
- Complete bipartite graph: A bipartite graph G with bipartition (A,B) is said to be complete bipartite if every vertex in A is adjacent to every vertex in B and vice versa.
- Cycle: A closed path is called a cycle.
- Labelling of a graph:
  Vertex labelling :It is a mapping from set of vertices to set of natural numbers .
  Edge labelling :It is a mapping from set of edges to set of natural numbers .
- Cordial labelling: For a given graph G label the vertices of G '0' or '1'. And every edge $'ab'$ of G will be labeled as '0' if the labeling of the vertices 'a' and 'b' are same and will be labeled as '1' if the labeling of the vertices 'a' and 'b' are different. Then this labeling is called a "cordial labeling" or the graph G is called a "cordial graph" iff,

$$|number\ of\ vertices\ labeled\ '0' - number\ of\ vertives\ labeled\ '1'| \leq 1$$
$$|number\ of\ edges\ labeled\ '0' - number\ of\ edges\ labeled\ '1'| \leq 1$$

Theorem 1. If G is a 1- regular bipartite graph with partite sets A and B with | A | = | B | = n Then G is cordial iff n = 0, 1, 3 mod 4.

Case I. n = 4m

Let m = 1 i.e. n = 4 then G looks like,



This can be labeled as shown is figure. This is CORDIAL.

Now for n = 4m for m > 1 'G' can be considered as combination of graph shown above and hence can be labeled repeatedly as above which is CORDIAL.

Hence,

G IS CORDIAL FOR n = 4m

Case II. n = 4m + 1

Consider $n = 4m + 1 \quad for\ m \in Z$

Consider any edge 'uv' $\in$ E (G), for $u \in A, v \in B$

Then G – {uv} is a bipartite graph with n = 4m. Hence it has a cordial labeling as given in case (i) which gives,

| # of edges labeled '0'- # of edges labeled '1'| = 0 hence along with the same labeling if we label "u" as '0' and "v" as '1' we get,

| # of vertices labeled '0'- # of vertices labeled '1'| = 0 and

| # of edges labeled '0'- # of edges labeled '1'| = 1

Hence it is cordial.

G IS CORDIAL FOR n = 4m+1

Case III. Let n = 4m + 2

Consider $n = 4m + 2 \quad for\ m \in Z$

|A| = |B|= 4m+2

Assume G is cordial.

Let α_0 be the number of vertices in A labeled '0'

Let α_1 be the number of vertices in A labeled '1'

Let β_0 be the number of vertices in B labeled '0'

Let β_1 be the number of vertices in B labeled '1'

$\therefore$ Total number of edges 'ab' where 'a' is labeled 0 and $b \in B = \alpha_0$

Total number of edges 'ab' where 'b' is labeled 0 and $a \in A = \beta_0$ -------*

Total number of edges 'ab' where 'a' is labeled 1 and $b \in B = \alpha_1$

Total number of edges 'ab' where 'b' is labeled 1 and $a \in A = \beta_1$

Let number of edges of type 0 – 0 (i.e. edge 'ab' where 'a' and 'b' both are labeled '0') be 'x'

And Let number of edges of type 1 – 1 be 'y'

From *

Number of edges of type 0 – 1 and 1 – 0 are, $\alpha_0 + \beta_0 - 2x$

Number of edges of type 0 – 1 and 1 – 0 are, $\alpha_1 + \beta_1 - 2y$

$\Rightarrow \alpha_0 + \beta_0 - 2x = \alpha_1 + \beta_1 - 2y$

But $\alpha_0 + \beta_0 = \alpha_1 + \beta_1$ By assumption G is cordial

$\Rightarrow$ x = y

But total number of edges = 4m+2

$\Rightarrow$ Number of edges labeled '0' = Number of edges labeled '1' $\because$ G is cordial

$\Rightarrow$ x + y = 2m+1

$\Rightarrow\ x = m + \frac{1}{2}\ and\ y = m + \frac{1}{2}$

Contradiction.

Hence 'G' is not cordial for n=4m+2

G IS NOT CORDIAL FOR n = 4m+2

Case IV Let n = 4m + 3

Consider $n = 4m + 3 \quad for\ m \in Z$

|A| = |B|= 4m+3

Let 'G' be a graph n = 4m+3

Consider any three edges $(u_1, v_1), (u_2, v_2), (u_3, v_3)$ with $u_i \in A and\ v_i \in B$ for i =1,2,3

Then G - $u_1, v_1, u_2, v_2, u_3, v_3$ is a graph with n = 4m

$\therefore$ By case i it has a cordial labeling with

| # of vertices labeled '0'- # of vertices labeled '1'| = 0 and
| # of edges labeled '0'- # of edges labeled '1'| = 0

Along with the same labeling label $u_1, v_1, u_2, v_2, u_3, v_3$ as follows,

$u_1\ as\ o, v_1\ as\ o,\ u_2\ as\ o,\ v_2\ as 1,\ u_3\ as 1,\ v_3\ as 1$

Then number of vertices labeled 0 = Then number of vertices labeled 1

And labeling of $(u_1, v_1)\ is\ 0, (u_2, v_2)\ is 1,\ (u_3, v_3)\ is\ 0$

Then number of vertices labeled 0 = Then number of vertices labeled '1'+1

| # of vertices labeled '0'- # of vertices labeled '1'| = 0 and
| # of edges labeled '0'- # of edges labeled '1'| = 1
$\therefore$ 'G' is cordial

G IS CORDIAL FOR n = 4m+3

Hence the result.

Theorem 2. If G is a 2- regular bipartite graph with partite sets A and B with | A | = | B | = n
Then G is cordial iff every component of G can be written as cycle of length 4m

Let 'G' be a bipartite regular graph of degree '2'.

We know that, " If 'G' is a regular bipartite graph of degree '2' then it can always be written as disjoint union of even cycles.

Let 'G' be the graph which is the cycle of length '2n'

Claim : Cycle of length 2n is cordial iff n is even.

Part (a) : To prove: n is even $\Rightarrow$ G is cordial.
Part (b) : To prove: G is cordial $\Rightarrow$ n is even.
i.e. To prove: n is odd $\Rightarrow$ G is not cordial.

Proof of (a): Consider a cycle of length m = 2n where n is even.
Let n = 2p $\Rightarrow$ m = 4p

Let $a_{1,} a_{2,} a_{3,} a_{4,} a_{5,} ......a_{4p,} a_{1,}$ be the given cycle where $a_{i,}$ is adjacent to $a_{i+1,}$

For $i = 1,2,3..........4p - 1$ and $a_{4p,}$ is adjacent to $a_{1,}$

Label $a_{1,} a_{2,} a_{3,} a_{4,} a_{5,} ......a_{4p,}$ as

0,1,1,0,0,1,1,0,0,1,1,…0,1,1,0

As m = 4p, we can have repeatedly p times the labeling of vertices as 0,1,1,0

Which gives edges labeling as,1,0,1,0,1,0,1,…0 [last edge will be labeled '0' as

$a_{4p,}$ and $a_{1,}$ both labeled 0.

We can see that the labeling is cordial.

Hence cycle of length '2n' where n is even is always cordial.

Proof of (b): Consider a cycle of length m = 2n where n is odd.

$Let\ n = 2p + 1 \Rightarrow m = 4p + 2$

We can write this graph as a bipartite graph with partite sets A and B where

| A | = | B | = 2p+1

Let number vertices of 'A' labeled '0' be α_0

Let number vertices of 'A' labeled '1' be α_1

Let number vertices of 'B' labeled '0' be β_0

Let number vertices of 'B' labeled '1' be β_1

Notation: $\alpha_{i\,j}$ --- number of edges 'ab' where

'a' is a vertex in A labeled 'i' and

'b' is a vertex in B labeled 'j'

With this notation we have,

$a_{11} + a_{10} = 2\alpha_1$ --------------- 1

$a_{00} + a_{01} = 2\alpha_0$ --------------- 2

$a_{10} + a_{00} = 2\beta_0$ --------------- 3

$a_{01} + a_{11} = 2\beta_1$ --------------- 4

Assuming 'G' is cordial we get,

$a_{11} + a_{00} = a_{10} + a_{01}$-------------- *

But, $a_{11} + a_{00} + a_{10} + a_{01} = 4p + 2$

$\therefore\ (a_{11} + a_{00}) + (a_{10} + a_{01}) = 4p + 2$ --------------**

From *and *

$(a_{11} + a_{00}) = 2p + 1$

$(a_{10} + a_{01}) = 2p + 1$

1 and 3 $\Rightarrow\ a_{11} + a_{00} = 2\alpha_1 + 2\beta_0 - 2a_{10}$

$\Rightarrow\ 2p + 1 = 2\alpha_1 + 2\beta_0 - 2a_{10}$

Which is a contradiction as L.H.S. is 1 mod 2 and R.H.S. is 0 mod 2

Hence this cycle cannot be cordial.

Hence a cycle of length 2n is not cordial if n is odd.

Thus we have proved: Cycle of length 2n is cordial iff n is even.

$\therefore$ A regular bipartite graph of degree 2 is cordial iff its every component can be written as a cycle of length 4n.

Hence the proof.

Theorem 3. If G is a n- regular bipartite graph with partite sets A and B with | A | = | B | = n Then G is cordial

Let $A = \{a_1, a_2, ...a_n\}$
$B = \{b_1, b_2, ...b_n\}$

Case I : n is even

Let n = 2m

Label $a_1, a_2, ...a_n$ as '0' and $a_{m+1}, a_{m+2}, ...a_{2m}$ as '1'.

For set B label any 'm' vertices as '0' any 'm' as '1'

W.l.g. let $b_1, b_2, ...b_n$ are '0' and $b_{m+1}, b_{m+2}, ...b_{2m}$ are '1'

For the vertex a_1 :The edges incident on a_1 are,

$a_1b_1, a_1b_2, ...a_1b_{m,}\ a_1b_{m+1}, a_1b_{m+2}, ...a_1b_{2m,}$ out of which,

$a_1b_1, a_1b_2, ...a_1b_m$ are labeled '0' and $a_1b_{m+1}, a_1b_{m+2}, ...a_1b_{2m,}$ are labeled '1'.

Which gives equal number of edges labeled 0 and 1 each equal to 'm'

Similarly for remaining vertices.

Hence we have in all $2m^2$ edges labeled o and $2m^2$ edges labeled 1.

$\therefore$ | # of edges labeled '0'- # of edges labeled '1'| = 1

$\therefore$ 'G' is cordial

Case II : n is odd

Let n = 2m+1

Consider A – { u } and B – { v } for some uv $\in$ V (G)

Then by case (i) it has a cordial labeling with

$2m^2$ edges labeled 0 and $2m^2$ edges labeled 1

Now label u as 0 and v as 1

As it is complete bipartite we have edges,

$ub_1, ub_2, ...ub_m, ub_{m+1}, ub_{m+2}, ...ub_{2m}, uv$ which are labeled as

0, 0, … 0, 1, 1, …. 1, 1 respectively giving m edges labeled 0 and m+1 edges labeled 1.

Also, $a_1v, a_2v, ...a_mv, a_{m+1}v, a_{m+2}v, ..., a_{2m}v$ which are labeled as

1, 1,… 1, 0, 0, …. 0 respectively giving m edges labeled 0 and m edges labeled 1.

$\therefore$ In all we get, Number of edges labeled 0 = $2m^2 + 2m$ and

Number of edges labeled 1 = $2m^2 + 2m + 1$

| # of edges labeled '0'- # of edges labeled '1'| = 1

$\therefore$ 'G' is cordial

Hence the proof.

Theorem 4. If G is a n- regular bipartite graph with partite sets A and B with | A | = | B | = n Then G is cordial iff n = 0, 1 mod 4.

Case (i) : n = 4m

Let A = { $a_1, a_2, ..., a_n$ } and B ={ $b_1, b_2, ...b_n$ }

And G is a (k-1) bipartite graph with partite sets A,B.

$\therefore$ It can be obtained by removing one and only one edge of every vertex of a complete graph.

Label $a_1, a_2, ...a_{2m}$ as 0 and $a_{2m+1}, a_{2m+2}, ...a_{4m}$ as 1

Then we can have 'G' is a graph obtained by removing

$a_1b_{i_1}, a_2b_{i_2}, ..., a_{4m}b_{i_{4m}}$ where $b_{i_j} \neq b_{i_k}$ *for* $i_j \neq i_k$ from the complete bipartite graph.

We will rename the vertices in the set 'B' such that the edges removes are,

$a_1b_1, a_2b_2, ...a_{4m}b_{4m}$

Now label $b_1, b_2, ...b_{4m}$ as,

$b_1, b_2, ...b_m$ as '0', $b_{m+1}, b_{m+2}, ...b_{2m}$ as '1', $b_{2m+1}, b_{2m+2}, ...b_{3m}$ as '0', $b_{3m+1}, b_{3m+2}, ...b_{4m}$ as '1',

Then the edges removed from the complete graph are labeled as,

$a_1b_1, a_2b_2, ...a_mb_m$ ________ 0,0,…0 ______ total 'm' edges

$a_{m+1}b_{m+1}, a_{m+2}b_{m+2}, ...a_{2m}b_{2m}$ ________ 1,1,…1 ______ total 'm' edges

$a_{2m+1}b_{2m+1}, a_{2m+2}b_{2m+2}, ...a_{3m}b_{3m}$ ________ 0,0,…0 ______ total 'm' edges

$a_{3m+1}b_{3m+1}, a_{3m+2}b_{3m+2}, ...a_{4m}b_{4m}$ ________ 1,1,…1 ______ total 'm' edges

$\therefore$ For 'G' Number of vertices labeled 0= number of vertices labeled 1= 4m

And Number of edges labeled 0= number of edges labeled 1= $8m^2 - 2m$

(from the case k = n)

$\therefore$ | # of vertices labeled '0'- # of vertices labeled '1'| = 0

and

| # of edges labeled '0'- # of edges labeled '1'| = 0

$\therefore$ 'G' is cordial

Case (ii) : n = 4m+1

As G is a (n-1)-regular bipartite graph , with the partitions A and B

Where A = { $a_1, a_2, ..., a_n$ } and B ={ $b_1, b_2, ...b_n$ }

By case (i) with the similar arguments let G is obtained from the complete bipartite graph by deleting the edges,

$a_1b_1, a_2b_2, ...a_{4m}b_{4m}, a_{4m+1}b_{4m+1}$ Where,

$a_1, a_2, ...a_{2m}, a_{2m+1}$ are labeled as 0 and $a_{2m+2}, a_{2m+2}, ...a_{4m+1}$ as 1.

Now label the vertices of B as follows.

$b_1, b_2, ...b_m$ as '0', $b_{m+1}, b_{m+2}, ...b_{2m}, b_{2m+1}$ as '1', $b_{2m+2}, b_{2m+2}, ...b_{3m+1}$ as '0', $b_{3m+2}, b_{3m+2}, ...b_{4m+1}$ as '1',

The edges removed from the complete graph and there labelings are as follows :

$a_1b_1, a_2b_2, ...a_mb_m$ ________ 0,0,…0 ______ total 'm' edges

$a_{m+1}b_{m+1}, a_{m+2}b_{m+2}, ...a_{2m}b_{2m}, a_{2m+1}b_{2m+1}$ ________ 1,1,…1 ______ total 'm+1' edges

$a_{2m+2}b_{2m+2}, a_{2m+3}b_{2m+3}, .., a_{3m+1}b_{3m+1}$ ________ 1,1,…1 ______ total 'm' edges

$a_{3m+2}b_{3m+2}, a_{3m+3}b_{3m+3}, ...a_{4m+1}b_{4m+1}$ ________ 0,0,…0 ______ total 'm' edges

$\therefore$ Number of edges labeled 1 are
$$8m^2 + 4m + 1 - (m+1) - m$$
$$= 8m^2 + 2m$$

Number of edges labeled 0 are
$$8m^2 + 4m - m - m$$
$$= 8m^2 + 2m$$

$\therefore$ Number of edges labeled 1= Number of edges labeled 0

Hence G is cordial.

(n-1) –regular graph is cordial for n = 4m+1

Case (iii) : n = 4m+2

As G is a (n-1)-regular bipartite graph, with the partitions A and B

Where A = { $a_1, a_2, ..., a_n$ } and B ={ $b_1, b_2, ...b_n$ }

$\therefore$ Number of edges in G is $(4m+2)(4m+1)=16m^2+12m+2$

Let α_0 be number of vertices in A labeled '0'

α_1 be number of vertices in A labeled '1'

β_0 be number of vertices in B labeled '0'

β_1 be number of vertices in B labeled '1'

$\therefore$ Total number of edges 'ab' where 'a' is labeled 0 and $b \in B = \alpha_0\,(4m+1)$

Total number of edges 'ab' where 'b' is labeled 0 and $a \in A = \beta_0\,(4m+1)$

Total number of edges 'ab' where 'a' is labeled 1 and $b \in B = \alpha_1\,(4m+1)$

Total number of edges 'ab' where 'b' is labeled 1 and $a \in A = \beta_1(4m+1)$

Let Total number of edges 'ab' where 'a' and 'b' both are labeled 0 =x

Total number of edges 'ab' where 'a' and 'b' both are labeled 1 =y

(i.e. 0-0 and 1-1 type)

$\therefore$ The number of edged 'ab' of labeling 0&1 And 1&0 are

$\alpha_0(4m+1)+\beta_0(4m+1)-x$ Also,

The number of edged 'ab' of labeling 1&0 And 0&01are

$\alpha_1(4m+1)+\beta_1(4m+1)-y$

$\therefore\ \alpha_0(4m+1)+\beta_0(4m+1)-x=\alpha_1(4m+1)+\beta_1(4m+1)-y$

$\Rightarrow(\alpha_0+\beta_0)(4m+1)-2x=(\alpha_1+\beta_1)(4m+1)-2y$

But, $(\alpha_0+\beta_0)=(\alpha_1+\beta_1)$ as G is cordial

$\Rightarrow$ x = y

Now ,as total number of edges are $16m^2+12m+2$

number of edges labeled 0 = $8m^2+6m+1$

$\Rightarrow$ x+y = $8m^2+6m+1 \Rightarrow$ 2x = $8m^2+6m+1 \quad \because x = y \text{ as } G \text{ is cordial}$

Contradiction $\because$ L.H.S. is even & R.H.S. is odd

Hence G is Not cordial.

(n-1) –regular graph is not cordial for n = 4m+2

Case (iii) : n = 4m+3

Total number of edges = $(4m+3)(4m+2)=16m^2+20m+6$

Using the same notations and the same arguments as in case (iii) we get,

$\therefore\ \alpha_0(4m+2)+\beta_0(4m+2)-x=\alpha_1(4m+2)+\beta_1(4m+2)-y$

$\Rightarrow(\alpha_0+\beta_0)(4m+2)-2x=(\alpha_1+\beta_1)(4m+2)-2y$

But, $(\alpha_0+\beta_0)=(\alpha_1+\beta_1)$ as G is cordial

$\Rightarrow$ x = y

Now ,as total number of edges are $16m^2+20m+6$

number of edges labeled 0 = $8m^2+10m+3$

$\Rightarrow$ x+y = $8m^2 + 10m + 3$

$\Rightarrow$ 2x = $8m^2 + 10m + 3 \quad \because x = y \; as \; G \; is \; cordial$

Contradiction $\because$ L.H.S. is even & R.H.S. is odd

Hence G is NOT cordial

(n-1) –regular graph is not cordial for n = 4m+3

Hence the proof.

II. Conclusion

- If G is a 1-regular bipartite graph with partite sets 'A' and 'B' such that | A | = | B | = n then 'G' is cordial iff n = 0 , 1 , 3 mod 4
- If G is a 2-regular bipartite graph with partite sets 'A' and 'B' such that | A | = | B | = n then 'G' is cordial iff its every component is a cycle of length 4m.
- If G is a n-regular $n \geq 3$ bipartite graph with partite sets 'A' and 'B' such that | A | = | B | = n then 'G' is cordial .
- If G is a $n-1$-regular bipartite graph with partite sets 'A' and 'B' such that | A | = | B | = n then 'G' is cordial iff $n = 0,1 \bmod 4$.

References

[1] L.CAI,X.ZHU, (2001), "Gaming coloring index of graphs", Journal of graph theory, 36, 144-155
[2] XUDING ZHU, (1999), "The game coloring number of planar graphs", Journal of Combinatorial Theory Series B, 75, 245-258
[3] H. YEH, XUDING ZHUING, (2003), "4-colorable, 6-regular toroidal graphs", Discrete Mathematics, 273, 1-3,261-274
[4] X.ZHU, (2006), "Recent development in circular coloring of graphs", Topics in discrete mathematics, spinger 5, 497-550
[5] H.CHANG, X.ZHU, (2008), "Coloring games on outerplanar graphs & trees", Discrete mathematics,doi10.1016/j.disc, 9-15
[6] H. HAJIABLOHASSAN, X. ZHU, (2003),: The circular chromatic number and mycielski construction", Journal of graph theory,404, 106-115