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On Hadamard Product of P-Valent Functions with Alternating Type

[1]S.S.Pokley, [2]K.D.Patil, [3]M.G.Shrigan
[1]Department of Mathematics, Kavikulguru Institute of Technology and Science, Ramtek-441106, (M.S.), India
[2]Department of Mathematics, Bapurao Deshmukh College of Engineering, Sevagram Wardha-442102(M.S.), India
[3]Department of Mathematics, G.H.Raisoni Institute of Engineering & Technology, Pune - 412207, (M.S.), India

Abstract: *Padmanabhan and Ganeshan have obtained some results on Hadamard product of univalent functions with negative coefficients of the type* $f(z) = z - \sum_{n=1}^{\infty} a_{2n} z^{2n}, a_{2n} \geq 0.$ *In this paper we have obtained coefficient bounds and convolution results of p-valent function.*
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I. Introduction

Let $\mathcal{A}$ (p) denote the class of *f normalized* univalent functions of the form

$$f(z) = z^p + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1} \quad , \; a_{n+1} \geq 0 \tag{1.1}$$

holomorphic & p-valent in the unit disc E = {z: z Є C;|z| < 1}.

A function f Є $\mathcal{A}$ (p) is said to belong to the class S (p, α) of p-valently starlike function of order α (0 ≤ α ≤ p) if it satisfies, for z ЄE, the condition

$$Re\left(\frac{z f'(z)}{f(z)}\right) > \alpha \tag{1.2}$$

A function f Є $\mathcal{A}$ (p) is said to belong to the class S (p, α) of p-valently convex functions of order α (0 ≤ α ≤ p) if it satisfies, for z ЄE, the condition

$$Re\left(1 + \frac{z f''(z)}{f'(z)}\right) > \alpha \tag{1.3}$$

The class S (p, α) was introduced by Goodman [1].

Silverman [2] studied the properties of these functions. Let K = {w/w is analytic in E; w (0) = 0, |w (z)| < 1 in E}. Let G (A, B) denote a subclass of analytic functions in E, which are of the form $\frac{1+Aw(z)}{1+Bw(z)}$, −1 ≤ A < B ≤ 1 where w (z) ∈ K.

Consider the following definitions,

$$S^*(A, B) = \left\{ f \,/\, f \in S \text{ and } \frac{z f'}{f} \in G(A, B) \right\}$$

$$H(A, B) = \left\{ f \,/\, f \in S \text{ and } \left(\frac{z f}{f'}\right)' \in G(A, B) \right\}$$

$$M^*(A, B) = \left\{ f \,/\, f \in M \text{ and } \frac{z f'}{f} \in G(A, B) \right\}$$

$$C(A,B)=\left\{ f\,/\,f\in S \text{ and } \left(\frac{z\,f'}{f'}\right)' \in G(A,B)\right\}$$

If $$f(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1} \quad , \; a_{n+1} \geq 0$$ and

$$g(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} b_{n+1} z^{n+1} \quad , \; b_{n+1} \geq 0$$

The class $\mathcal{A}$ (p) is closed under the Hardmard product (or Convolution) of f & g as,

$$h(z)=f(z)*g(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1}\, b_{n+1} z^{n+1} \quad , a_{n+1}\, , b_{n+1} \geq 0 \qquad (1.4).$$

K..S. Padmanabhan and M.S. Ganeshan [3] also S.M.Khainar and Meena More [4] studied some convolution properties of functions with negative coefficients. In this paper we extend the results on convolution property for p-valent function in the class M^{*} (A, B) and C (A, B).

II. Preliminary and Main Results

We start with lemma which will be required for further investigation convolution results.

Leema 2.1 : *A function*

$$f(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1} \quad , \; a_{n+1} \geq 0$$

is in $M^{*}(A,B)$ *iff* ,

$$\left(\frac{(p-1)+\sum_{n=p}^{\infty}\left[n(B+1)-(A-B)\right]a_{n+1}}{(A-pB)}\right) \leq 1.$$

Proof : We have

$$f(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1} \quad , \; a_{n+1} \geq 0$$

then

$$\frac{z f'(z)}{f(z)} = \frac{p\,z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1)\, z^{n+1}}{z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1}} \in G(A,B)$$

iff

$$\frac{p\,z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1)\, z^{n+1}}{z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1}} = \frac{1+A\,w(z)}{1+B\,w(z)}$$

On Simplification, we get

$$w(z)\left\{(A-pB)z^p+\sum_{n=p}^{\infty}[A-B(n+1)](-1)^{n+1}a_{n+1}z^{n+1}\right\}=(p-1)z^p+\sum_{n=p}^{\infty}n(-1)^{n+1}a_{n+1}z^{n+1}$$

Therefore

$$w(z)=\frac{(p-1)z^p+\sum_{n=p}^{\infty}n(-1)^{n+1}a_{n+1}z^{n+1}}{\left\{(A-pB)z^p+\sum_{n=p}^{\infty}[A-B(n+1)](-1)^{n+1}a_{n+1}z^{n+1}\right\}}$$

Notice that $|w(z)|\le 1$, we have

$$\left|\frac{(p-1)z^p+\sum_{n=p}^{\infty}n(-1)^{n+1}a_{n+1}z^{n+1}}{\left\{(A-pB)z^p+\sum_{n=p}^{\infty}[A-B(n+1)](-1)^{n+1}a_{n+1}z^{n+1}\right\}}\right|\le 1$$

Allowing $|z|=r\to 1$, we get

$$\frac{(p-1)+\sum_{n=p}^{\infty}n a_{n+1}}{(A-pB)+\sum_{n=p}^{\infty}[A-B(n+1)]\ a_{n+1}}\le 1$$

Therefore

$$(p-1)+\sum_{n=p}^{\infty}[n(B+1)-(A-B)]\ a_{n+1}\le A-pB$$

$$\left(\frac{(p-1)+\sum_{n=p}^{\infty}[n(B+1)-(A-B)]a_{n+1}}{(A-pB)}\right)\le 1$$

Leema 2.**2** : *A function*

$$f(z)=z^p+\sum_{n=p}^{\infty}(-1)^{n+1}a_{n+1}z^{n+1}\quad ,\ a_{n+1}\ge 0$$

is in c(A,B) iff,

$$\left(\frac{(p^2-p)+\sum_{n=p}^{\infty}(n+1)[n(B+1)-(A-B)]a_{n+1}}{p(A-pB)}\right)\le 1.$$

Proof : We have

$$f(z)=z^p+\sum_{n=p}^{\infty}(-1)^{n+1}a_{n+1}z^{n+1}\quad ,\ a_{n+1}\ \ge\ 0$$

then

$$\frac{\left(z f^{'}(z)\right)^{'}}{f^{'}(z)} = \frac{p^2 z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1)^2 z^n}{p z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1) z^n} \in G(A,B)$$

iff

$$\frac{p^2 z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1)^2 z^n}{p z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} (n+1) z^n} = \frac{1 + A\,w(z)}{1 + B\,w(z)}$$

$$w(z)\left\{(PA - P^2B)z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1}\left[A(n+1) - B(n+1)^2\right] a_{n+1} z^n\right\}$$

$$= (p^2 - p) z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} [n(n+1)] a_{n+1} z^n$$

Again since $|w(z)| \le 1$, we get

$$\left|\frac{(p^2 - p) z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1} [n(n+1)] a_{n+1} z^n}{(PA - P^2B)z^{p-1} + \sum_{n=p}^{\infty} (-1)^{n+1}\left[A(n+1) - B(n+1)^2\right] a_{n+1} z^n}\right| \le 1$$

Allowing $|z| = r \to 1$ we get

$$\left|\frac{p(p-1) + \sum_{n=p}^{\infty} [n(n+1)] a_{n+1} z^n}{p(A - PB) + \sum_{n=p}^{\infty}\left[A(n+1) - B(n+1)^2\right] a_{n+1}}\right| \le 1$$

$$\left|\frac{p(p-1) + \sum_{n=p}^{\infty} [n(n+1)] a_{n+1} z^n}{p(A\text{-}PB) + \sum_{n=p}^{\infty}\left[A(n+1) - B(n+1)^2\right] a_{n+1}}\right| \le 1$$

$$(p^2 - p) + \sum_{n=p}^{\infty} (n+1)[(B+1)n - (A-B)]\, a_{n+1} \le p(A - PB)$$

$$\frac{(p^2 - p) + \sum_{n=p}^{\infty} (n+1)[(B+1)n - (A-B)]\, a_{n+1}}{p(A - PB)} \le 1$$

We define

$$h(z) = f(z) * g(z) = z^p + \sum_{n=p}^{\infty} (-1)^n a_{n+1} b_{n+1} z^{n+1} \quad , a_{n+1}, b_{n+1} \ge 0$$

for $f(z)$ and $g(z)$ members of $M^*(A,B)$ and $C(A,B)$.

Theroem 2.1 *: A function*

$$f(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} a_{n+1} z^{n+1} \quad , a_{n+1} \geq 0$$

$$g(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n+1} b_{n+1} z^{n+1} \quad , b_{n+1} \geq 0$$

are elements of class $M^{*}(A,B)$ *then*

$$h(z)= f(z)*g(z)= z^{p} + \sum_{n=p}^{\infty} (-1)^{n} a_{n+1}\, b_{n+1} z^{n+1} \quad ,a_{n+1} ,b_{n+1} \geq 0$$

is element of $M^{*}(A,B)$ *with* $-1 \leq A_1 \leq B_1 \leq 1$, *where* $A_1 \geq -1, B_1 \leq \frac{A_1-2K}{1+2K}$

these bounds for A_1 *and* B_1 *are sharp.*

Proof : *By lemma* 1, We have

$$\left(\frac{(p-1)+ \sum_{n=p}^{\infty}\left[n(B+1)-(A-B)\right]a_{n+1}}{(A-pB)} \right) \leq 1$$

Since $f \in \mathrm{M}^{*}(\mathrm{A,B})$. (2.1)

$$\left(\frac{(p-1)+ \sum_{n=p}^{\infty}\left[n(B+1)-(A-B)\right]b_{n+1}}{(A-pB)} \right) \leq 1$$

Since $g \in \mathrm{M}^{*}(\mathrm{A,B})$. (2.2)

We want to find A_1 & B_1 *such that* $-1 \leq A_1 \leq B_1 \leq 1$, *for* $h(z) = f(z)*g(z) \in \mathrm{M}^{*}(\mathrm{A,B})$.

Now $h(z) \in \mathrm{M}^{*}(\mathrm{A,B})$ if

$$\left(\frac{(p-1)+\sum_{n=p}^{\infty}\left[n(B_1+1)-(A_1-B_1)\right]a_{n+1}b_{n+1}}{(A_1-pB_1)} \right) \le 1 \ ; by\ lemma\ 2.1 \tag{2.3}$$

$$\left(\frac{(p-1)}{(A_1-pB_1)}+\sum_{n=p}^{\infty} \frac{\left[n(B_1+1)-(A_1-B_1)\right]a_{n+1}b_{n+1}}{(A_1-pB_1)} \right) \le 1$$

$$\frac{(p-1)}{(A_1-pB_1)}+\sum_{n=p}^{\infty} u_1 a_{n+1}b_{n+1} \le 1$$

$$where\ u_1=\frac{\left[n(B_1+1)-(A_1-B_1)\right]}{(A_1-pB_1)}$$

By Cauchy Schwarz inequlity,

$$\sum_{n=p}^{\infty}\sqrt{ua_{n+1}b_{n+1}} \le \left(\sum_{n=p}^{\infty} ua_{n+1}\right)^{\frac{1}{2}} \left(\sum_{n=p}^{\infty} ub_{n+1}\right)^{\frac{1}{2}} \le 1 \tag{2.4}$$

$$where\ u=\frac{\left[n(B+1)-(A-B)\right]}{(A-pB)} \tag{2.5}$$

(1.3) *is true if*

$$\frac{(p-1)}{(A_1-pB_1)}+u_1a_{n+1}b_{n+1} \le \frac{(p-1)}{(A-pB)}+u\sqrt{a_{n+1}b_{n+1}}$$

If $A_1=A$ *and* $B_1=B$ *then we get,*

$$u_1a_{n+1}b_{n+1} \le u\sqrt{a_{n+1}b_{n+1}} \tag{2.6}$$

i.e. $$u_1\sqrt{a_{n+1}b_{n+1}} \le u$$

Therefore it is enough to find u such that

$$\frac{1}{u} \le \frac{u}{u_1} \quad i.e. \quad u_1 \le u^2 \tag{2.7}$$

On Simplification, we get

$$\frac{[(p-1)+n(B_1+1)-(A_1-B_1)]}{(A_1-pB_1)} \le \left\{\frac{[(p-1)+n(B+1)-(A-B)]}{(A-pB)}\right\}^2 = u^2$$

$$\frac{[(p-1)+n(B_1+1)-(A_1-B_1)]}{(A_1-pB_1)} \le u^2$$

i.e. $(p-1)+n(B_1+1)-(A_1-B_1) \le (A_1-pB_1)\, u^2$

$$A_1 \ge \frac{(p-1)+(B_1+1)n+B_1(pu^2+1)}{(1+u^2)} \qquad (2.8)$$

Taking $B_1=1$ *and* $n=p$ *in* (1.8) *and* (1.5) *above we get*

$$A_1 \ge \frac{(p-1)+2p+(pu^2+1)}{(1+u^2)}$$

$$A_1 \ge \frac{p(3+u^2)}{(u^2+1)}$$

$$\ge p\left[1+2\frac{(A-pB)^2}{(A-pB)^2+\left[(p-1)+p(B+1)-(A-B)^2\right]}\right]$$

$$A_1 \ge p\,(1+2k)$$

where $$k=\frac{(A-pB)^2}{(A-pB)^2+\left[(p-1)+p(B+1)-(A-B)^2\right]} \qquad (2.9)$$

Theroem 2.2: *If* $f(z)\in M^*(A,B)$ *and* $g(z)\in M^*(A',B')$ *then* $f(z)*g(z)\in M^*(A_1,B_1)$ *where* $A_1\ge 1, B_1 \le \frac{A_1-2k}{1+2k}$ *with*

$$k=\frac{(A-pB)(A'-pB')}{(3B_1+p-A_1+1)(3B'+p-A'+1)+[(A_1-B_1)-(P-1)](A'-PB')} \qquad (2.10)$$

Proof : Pr*oceeding with the* arg*ument developed in Theroem* 1, *we require*

$$\frac{[(p-1)+n(B_1+1)-(A_1-B_1)]}{(A_1-pB_1)} \le \frac{[(p-1)+n(B_1+1)-(A_1-B_1)]}{(A_1-pB_1)}\frac{[(p-1)+n(B'+1)-(A'-B')]}{(A'-pB')} = \alpha$$

That is

$$\frac{(p-1)+n(B_1+1)}{(A_1-pB_1)} \le \alpha + \frac{A_1-B_1}{A_1-pB_1}$$

$$\frac{A_1-pB_1}{B_1+1} \ge \frac{n(A_1-pB_1)}{\alpha(A_1-pB_1)+(A_1-B_1)-(p-1)}$$

Notice that $\frac{n(A_1-pB_1)}{\alpha(A_1-pB_1)+(A_1-B_1)-(p-1)}$ *decreases as n increases.Simplifying, we get*

$$\frac{A_1 - pB_1}{B_1 + 1} \geq \frac{n(A - pB)(A' - pB')}{(3B_1 + p - A_1 + 1)(3B' + p - A' + 1) + [(A_1 - B_1) - (P-1)](A' - PB')} \qquad (2.11)$$

Taking $n = 2$, *we get*

$$\frac{A_1 - pB_1}{B_1 + 1} \geq \frac{2(A - pB)(A' - pB')}{(3B_1 + p - A_1 + 1)(3B' + p - A' + 1) + [(A_1 - B_1) - (P-1)](A' - PB')}$$

i.e. $\frac{A_1 - pB_1}{B_1 + 1} \geq 2k \qquad (2.12)$

where $k = \frac{n(A - pB)(A' - pB')}{(3B_1 + p - A_1 + 1)(3B' + p - A' + 1) + [(A_1 - B_1) - (P-1)](A' - PB')}$

Theroem2.3: *If* $f(z) \in C(A,B)$ *and* $g(z) \in C(A',B')$ *then* $f(z) * g(z) \in C(A_1, B_1)$ *where*

$A_1 \geq 1, B_1 \leq \frac{A_1 - 2k}{1 + 2k}$ *with*

$$k = \frac{(A - pB)(A' - pB')}{9(3B_1 + p - A_1 + 1)(3B' + p - A' + 1) + [(A_1 - B_1) - (P-1)](A' - PB')} \qquad (2.13)$$

Proof : *Let us verify with the following example*

$$f(z) = z^p - \frac{(A - pB)}{3(3B + p - A + 2)} z^{P+2} \in C(A, B)$$

$$g(z) = z^p - \frac{(A' - pB')}{3(3B' + p - A' + 2)} z^{p+2} \in C(A', B')$$

Then

$$f(z) * g(z) = z^p - \frac{(A - pB)(A' - pB')}{9(3B + p - A + 2)(3B' + p - A' + 2)} z^{p+1} \in C(A_1, B_1) \qquad (2.14)$$

if

$$\frac{(p-1) + n(B_1 + 1) - (A_1 + B_1)}{A_1 - pB_1} = \frac{9(3B + p - A + 2)(3B' + p - A' + 2)}{(A - pB)(A' - pB')}$$

Simplifying, we get

$$A_1 = p(1 + 2K)$$

where $K = \frac{(\mathrm{A} - \mathrm{pB})(\mathrm{A}' - \mathrm{pB}')}{9(3\mathrm{B}_1 + \mathrm{p} - \mathrm{A}_1 + 1)(3\mathrm{B}' + \mathrm{p} - \mathrm{A}' + 1) + [(\mathrm{A}_1 - \mathrm{B}_1) - (\mathrm{P} - 1)](\mathrm{A}' - \mathrm{PB}')}$

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