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Convolution Theorem for Canonical Cosine Transform and Their Properties

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Abstract: *The Canonical Cosine transform, which is a generalization of the linear canonical transform, has many applications in several areas, including signal processing and optics. In this paper we have introduced convolution theorem, linearity property, derivative property, modulation property and Parseval's identity for the generalized canonical cosine transform.*
Keywords: *Fractional Fourier Transform, Linear Canonical Transform, Convolution.*

I. Introduction:

As generalization of the Fourier Transform (FT), the Fractional Fourier Transform (FrFT) has been used in several areas, including optics and signal processing. Many properties for this transform are already known. In recent years, Almeida [1] and Zayed [4] derived product and convolution of two functions in a usual manner and proved the convolution theorem in fractional Fourier transform domain. In the past decade, FRFT has attracted much attention of the signal processing community, as the generalization of FT. The relevant theory has been developed including uncertainty principle, sampling theory, convolution theorem.

A further generalization of FrFT is Linear Canonical Transform (LCT). Just as Fourier cosine transform and Fourier sine transform are defined from Fourier Transform, similarly canonical cosine and canonical sine transforms are defined from LCT by Pie and Ding [3]. We have discussed some properties of Half canonical cosine transform in [2].

This paper emphasizes on defining generalized canonical cosine transform and deriving its convolution theorem, then some properties of the canonical cosine transform are discussed and finally conclusions are given.

Notations and terminology as per [5]

II. Testing Function Space $\mathcal{E}$:

An infinitely differentiable complex valued function ϕ on R^n belongs to $\mathcal{E}$ (R^n), if for each compact set, $I \subset S_\alpha$ where $S_\alpha = \{t : t \in R^n, |t| \le \alpha, \alpha > 0\}$ and for $k \in R^n$,

$$\gamma_{\mathcal{E},k}\,\varphi(t) = \sup_{t \in I}\left|D^k \varphi(t)\right| < \infty.$$

Note that space E is complete and a Frechet space, let E' denotes the dual space of E.

2.1Definition: The generalized Canonical Cosine Transform $f \in \mathcal{E}'(R^n)$ can be defined by,

{CCT $f(t)$} (s) = < $f(t)$, $K_C(t, s)$ > where,

$$K_C(t,s) = \sqrt{\frac{1}{2\pi.ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cdot \cos\left(\frac{s}{b}t\right)$$

Hence the generalized canonical cosine transform of a regular function $f \in \mathcal{E}'(R^n)$ can be defined by,

$$\{\, CCTf(t) \,\}(s) = \sqrt{\frac{1}{2\pi ib}} \cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_{-\infty}^{\infty} \cos\left(\frac{s}{b}t\right) \cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f(t)\, dt \qquad (1.1)$$

2.2 The Generalized Canonical Cosine Transform of Convolution

Now we introduced a special type of convolution and product for canonical cosine transform.

2.3 Definition: For any function $f(x)$, let us define the functions $\tilde{f}\ (x)$ and $\tilde{g}\ (x)$ by

$$\tilde{g}(|v-y|) = e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} g(v-y), \quad \tilde{g}(v+y) = e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} g(v+y)\} \text{ and } \tilde{f}(y) = e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} f(y)$$

For any two functions f and g, we define the Convolution operation $\star$ by

$$h(v) = (f \star g)(v) = \int_0^\infty \tilde{f}(y)(\tilde{g}(y+v) + \tilde{g}(|v-y|))dy \tag{1.2}$$

Now we state and prove convolution theorem.

3.1 Convolution Theorem:

If $h(s) = (f * g)(s)$ and F_C , G_C and H_C denote the Canonical Cosine transform of f, g and h respectively, then

$$H_C(s) = [F_C(f(y))](s)[G_C(g(t))](s) = \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{\sqrt{2\pi i b}} \left\{F_C\left(\tilde{f} * \tilde{g}\right)(v)\right\}(s)$$

Proof: From the definition of the Canonical Cosine transform, we have $H_C(s) = [F_C(f(y))](s)[G_C(g(t))](s)$

$$[F_C(f(y))](s)[G_C(g(t))](s) = \sqrt{\frac{1}{2\pi i b}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} \cos\left(\frac{s}{b}y\right) f(y)\,dy \sqrt{\frac{1}{2\pi i b}}\, e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2} \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} \cos\left(\frac{s}{b}t\right) g(t)\,dt$$

$$= \frac{1}{2i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} 2\cos\left(\frac{s}{b}y\right)\cos\left(\frac{s}{b}t\right) f(y)\,g(t)\,dy\,dt$$

$$= \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \left\{\cos\left(\frac{s}{b}(y+t)\right) + \cos\left(\frac{s}{b}(y-t)\right)\right\} f(y)\,g(t)\,dy\,dt$$

$$= \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \cos\left(\frac{s}{b}(y+t)\right) f(y)\,g(t)\,dy\,dt +$$

$$\frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_0^\infty \int_0^\infty e^{\frac{i}{2}\left(\frac{a}{b}\right)(y^2+t^2)} \cos\left(\frac{s}{b}(y-t)\right) f(y)\,g(t)\,dy\,dt$$

$$= I_1 + I_2 \tag{1.3}$$

For I_1, putting $y + t = v \Rightarrow dt = dv$ for limit when $t = o \Rightarrow v = y$, when $t = \infty \Rightarrow v = \infty$

For I_2, putting $y - t = -v \Rightarrow dt = dv$ for limit when $t = o \Rightarrow v = -y$, when $t = \infty \Rightarrow v = \infty$

$$(1.3) \Rightarrow [F_C(f(y))](s)[G_C(g(t))](s) = \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=y}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\,g(v-y)\,dy\,dv +$$

$$\frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=-y}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}(-v)\right) f(y)\,g(v+y)\,dy\,dv$$

$$[F_C(f(y))](s)[G_C(g(t))](s) = \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\,g(v-y)\,dy\,dv$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=y}^{0} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2} \cos\left(\frac{s}{b}v\right) f(y)\,g(v-y)\,dy\,dv$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=0}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\,g(v+y)\,dy\,dv +$$

$$+ \frac{1}{i\pi b} e^{i\left(\frac{d}{b}\right)s^2} \int_{y=0}^{\infty} \int_{v=-y}^{0} e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2} e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2} \cos\left(\frac{s}{b}v\right) f(y)\,g(v+y)\,dy\,dv$$

$$[F_C(f(y))](s)[G_C(g(t))](s)=\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$-\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{y}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$+\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v+y)\,dy\,dv+$$

$$+\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=-y}^{0}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v+y)\,dy\,dv$$

$$[F_C(f(y))](s)[G_C(g(t))](s)=I_1+I_2+I_3+I_4 \quad (1.4)$$

For I_4, putting $v=-v\Rightarrow dv=-dv$ for limit when $v=-y\Rightarrow v=y$, when $v=0\Rightarrow v=0$

$$I_4=\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=y}^{0}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y-v)^2}\cos\left(\frac{s}{b}(-v)\right)f(y)\,g(y-v)\,dy\,(-dv)$$

$$=\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{y}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y-v)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(y-v)\,dy\,dv$$

$$(1.4)\Rightarrow[F_C(f(y))](s)[G_C(g(t))](s)=\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$-\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{y}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$+\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v+y)\,dy\,dv$$

$$+\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{y}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$[F_C(f(y))](s)[G_C(g(t))](s)=\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v-y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v-y)\,dy\,dv$$

$$+\frac{1}{i\pi b}e^{i\left(\frac{d}{b}\right)s^2}\int_{y=0}^{\infty}\int_{v=0}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)y^2}e^{\frac{i}{2}\left(\frac{a}{b}\right)(v+y)^2}\cos\left(\frac{s}{b}v\right)f(y)\,g(v+y)\,dy\,dv$$

$$=\frac{1}{\sqrt{2i\pi b}}\frac{\sqrt{2}}{\sqrt{i\pi b}}e^{i\left(\frac{d}{b}\right)s^2}e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}\int_0^{\infty}\left\{\int_0^{\infty}\tilde{f}(y)\left(\tilde{g}(y+v)+\tilde{g}(|v-y|)\right)dy\right\}\cos\left(\frac{s}{b}v\right)e^{\frac{i}{2}\left(\frac{a}{b}\right)v^2}\,dv$$

by (1.2)

$$=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{\sqrt{2\pi ib}}\left\{\sqrt{\frac{2}{i\pi b}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_0^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)v^2}\cos\left(\frac{s}{b}v\right)\{(\tilde{f}*\tilde{g})(v)\}\,dv\right\}$$

by (2.3)

$$[F_C(f(y))](s)[G_C(g(t))](s)=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}e^{-\frac{i}{2}\left(\frac{a}{b}\right)v^2}}{\sqrt{2\pi ib}}\left\{F_C\left(\tilde{f}*\tilde{g}\right)(v)\right\}(s)$$

4.1 Some operational results:

4.1.1 Linearity property of canonical cosine transformations:

If $\{CCT\, f(t)\}(s)$, $\{CCT\, g(t)\}(s)$ denotes generalized canonical cosine transform of *f(t), g(t)* and P_1, P_2 are constants then $\{CCT\,(P_1 f(t)+P_2 g(t))\}(s)=P_1\{CCT\,(f(t)\}(s)+P_2\{CCT\,(g(t))\}(s)$

Proof is simple and hence omitted.

4.1.2 Derivative (with respect to parameter) of canonical cosine transform:

If *{CCT f(t)}(s)* denotes generalized canonical cosine transform, then,

$$\frac{d}{ds}\left[\{CCTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{CCTf(t)\}(s)-\frac{1}{b}\{CST[t.f(t)]\}(s)\right\}$$

Proof: We have,

$$\frac{d}{ds}\{CCT\ f(t)\}(s)=\frac{d}{ds}\left\{\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right\}$$

$$\frac{d}{ds}\{CCT\ f(t)\}(s)=\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\frac{\partial}{\partial s}\left(e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\cos\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(-\left(\frac{t}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)+i\left(\frac{d}{b}\right).s.e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\right)f(t)dt$$

$$=i\left\{-(-i)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\left(\frac{1}{b}\right)e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}.\sin\left(\frac{s}{b}t\right)[t.f(t)]dt+s.\left(\frac{d}{b}\right)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right\}$$

$$=i\{-\frac{1}{b}[CST[t.f(t)]\}(s)+s.\left(\frac{d}{b}\right)\{CCTf(t)\}(s)$$

$$\frac{d}{ds}\left[\{CCTf(t)\}(s)\right]=i\left\{s.\left(\frac{d}{b}\right)\{CCTf(t)\}(s)-\frac{1}{b}\{CST[t.f(t)]\}(s)\right\}$$

4.1.3 Modulation property of canonical cosine transform:

If *{CCT f(t)} (s)* denotes generalized canonical cosine transform of *f(t) t*hen,

$$\{CCT\cos zt.f(t)\}(s)=\frac{e^{-\frac{i}{2}dbz^2}}{2}\left\{\left[CCT\ f(t)\cdot e^{idsz}\right](\frac{s+bz}{b})+\left[CCT\ f(t)\cdot e^{-idsz}\right](\frac{s-bz}{b})\right\}$$

Proof: By definition of CCT,

$$\therefore\{CCT\ \cos zt.f(t)\}(s)=\frac{1}{2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\left(\cos\left(\frac{s}{b}+z\right)t+\cos\left(\frac{s}{b}-z\right)t\right)f(t)\,dt$$

$$\therefore\{CCT\ \cos zt.f(t)\}(s)=\frac{1}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}e^{-idsz}e^{-\frac{i}{2}(db)z^2}\cos\left(\frac{s}{b}+z\right)t.\ f(t)\,dt+$$
$$\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}e^{idsz}e^{-\frac{i}{2}(db)z^2}\cos\left(\frac{s}{b}-z\right)t..\ f(t)\,dt\}$$

$$\therefore\{CCT\ \cos zt.f(t)\}(s)=\frac{e^{-\frac{i}{2}(db)z^2}}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\cos\left(\frac{(s+bz)}{b}\right)t.e^{-idsz}\ f(t)\,dt+$$
$$\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\cos\left(\frac{(s-bz)}{b}\right)t.e^{idsz}.f(t)\,dt\}$$

$$\{CCT\cos zt.\,f(t)\}(s)=\frac{e^{-\frac{i}{2}dbz^2}}{2}\left\{[CCT\ f(t)\cdot e^{idsz}](\frac{s+bz}{b})+[CCT\ f(t)\cdot e^{-idsz}](\frac{s-bz}{b})\right\}$$

4.1.4 If *{CCT f(t)} (s)* denotes generalized canonical cosine transform of *f(t)* then,

$$\{CCT\sin zt.\,f(t)\}(s)=(-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{[CST\ f(t)\cdot e^{-idsz}](\frac{s+bz}{b})-[CCT\ f(t)\cdot e^{idsz}](\frac{s-bz}{b})\right\}$$

Proof: By definition of CCT,

$$\therefore\{CCT\ \sin zt.f(t)\}(s)=\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\cos\left(\frac{s}{b}t\right)\cdot\sin zt\cdot f(t)\,dt$$

$$\therefore\{CCT\ \sin zt.f(t)\}(s)=\frac{1}{2}\sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}.\left(\sin\left(\frac{s}{b}+z\right)t-\sin\left(\frac{s}{b}-z\right)t\right)f(t)\,dt$$

$$\therefore\{CCT\ \sin zt.f(t)\}(s)=\frac{1}{2}\{\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}e^{-idsz}e^{-\frac{i}{2}(db)z^2}\sin\left(\frac{s}{b}+z\right)t.\,f(t)\,dt$$

$$-\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}e^{idsz}e^{-\frac{i}{2}(db)z^2}\sin\left(\frac{s}{b}-z\right)t..\,f(t)\,dt\}$$

$$\therefore\{CCT\ \sin zt.f(t)\}(s)=\frac{(-i)e^{-\frac{i}{2}(db)z^2}}{2}\{(-i)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s+bz)^2}\sin\left(\frac{s+bz}{b}\right)t.e^{-idsz}f(t)\,dt$$

$$-(-i)\sqrt{\frac{1}{2\pi ib}}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}e^{\frac{i}{2}\left(\frac{d}{b}\right)(s-bz)^2}\cos\left(\frac{s-bz}{b}\right)t.e^{idsz}.\,f(t)\,dt\}$$

$$\{CCT\sin zt.f(t)\}(s)=(-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{[CST\ f(t)\cdot e^{-idsz}](\frac{s+bz}{b})-[CCT\ f(t)\cdot e^{idsz}](\frac{s-bz}{b})\right\}$$

5.1 Parseval's Identity for canonical cosine transform:

If $f(t)$ and $g(t)$ are the inversion canonical cosine transform of $F_C(s)$ and $G_C(s)$ respectively, then

(1) $\int_{-\infty}^{\infty}f(t).\overline{g(t)}\,dt=-2\pi\int_{-\infty}^{\infty}F_C(s).\overline{G_C(s)}\,ds$ and (2) $\int_{-\infty}^{\infty}.|f(t)|^2\,dt=-2\pi\int_{-\infty}^{\infty}|F_C(s)|^2\,ds$

Proof: By definition of CCT,

$$\{CCT\ g(t)\}(s)=\sqrt{\frac{1}{2\pi ib}}\cdot e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\cos\left(\frac{s}{b}t\right)\cdot g(t)\,dt$$

---------------- by (1.1)

Using the inversion formula of CCT

$$g(t)=\sqrt{\frac{2\pi i}{b}}\cdot e^{-\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{-\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)G_C(s)\,ds$$

Taking complex conjugate we get,

$$\overline{g(t)}=\sqrt{\frac{-2\pi i}{b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\overline{G_C(s)}\,ds$$

$$\int_{-\infty}^{\infty} f(t).\overline{g(t)}\,dt = \int_{-\infty}^{\infty} f(t)dt\left(\sqrt{\frac{-2\pi i}{b}}\cdot e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)\overline{G_C(s)}\,ds\right)$$

Changing the order of integration, we get,

$$\int_{-\infty}^{\infty} f(t).\overline{g(t)}\,dt = \sqrt{\frac{-2\pi i}{b}}\int_{-\infty}^{\infty}\overline{G_C(s)}\,ds\,\frac{1}{\sqrt{\frac{1}{2\pi ib}}}\left(\sqrt{\frac{1}{2\pi ib}}\;e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{d}{b}\right)s^2}\cos\left(\frac{s}{b}t\right)f(t)dt\right) \qquad (1.5)$$

(2) Now putting $f(t) = g(t)$ in equation (1.5), we get

$$\int_{-\infty}^{\infty} \left|f(t)\right|^2 dt = -2\pi\int_{-\infty}^{\infty}\left|F_C(s)\right|^2 ds$$

Table for canonical cosine transform

S.N.	*f(t)*	$F_C(s)$
1	$(P_1 f(t) + P_2 g(t))$	$P_1\{CCT\ (f(t)\}(s) + P_2\{CCT\ (g(t))\}(s)$
2	$\cos zt.f(t)$	$\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CCT\ f(t)\cdot e^{idsz}\right](\frac{s+bz}{b}) + \left[CCT\ f(t)\cdot e^{-idsz}\right](\frac{s-bz}{b})\right\}$
3	$\sin zt.f(t)$	$(-i)\frac{e^{-\frac{i}{2}(db)z^2}}{2}\left\{\left[CST\ f(t)\cdot e^{-idsz}\right](\frac{s+bz}{b}) - \left[CCT\ f(t)\cdot e^{idsz}\right](\frac{s-bz}{b})\right\}$

III. Conclusion:

The Convolution of generalized canonical cosine transform is developed in this paper. Operation transform formulae proved in this paper can be used, when this transform is used to solve ordinary or partial differential equation.

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