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Congruence Lattices of Isoform Lattices

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Abstract: *A congruence θ of a lattice L is said to be isoform, if any two congruence classes of θ are isomorphic as lattices. The lattice L is said to be isoform, if all congruence's of L are isoform. We prove that every finite distributive lattice D can be represented as the congruence lattice of a finite isoform lattice.*

I. Introduction

In this chapter we study about finite lattices with isoform congruences. We prove that every finite distributive lattice D can be represented as the congruence lattice of a finite isoform lattice. Infact, we prove that every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L with the following properties :

(i) L is isoform
(ii) For every congruence θ of L, the congruence classes of θ are projective intervals.
(iii) L is a finite pruned Boolean lattice.
(iv) L is discrete –transitive.

This result is a stronger version of the result obtained in the previous chapter.

To prove this result, we introduce a new lattice construction which is described in section 1.2. The congruence structure of this new construct is described section 1.3. In section 1.4, we present the proof of the main theorem.

We start with the definition of isoform lattices.

DEFINITION : 1.1.1

Let L be a lattice. Let θ be a congruence of L. Then θ is said to be isoform, if any two congruence classes of θ are isomorphic as lattices.

DEFINITION : 1.1.2

A lattice L is said to be isoform if all congruences of L are isoform.

DEFINITION : 1.1.3

A lattice L is said to be regular, if whenever two congruences share a congruence class, then the congruences are the same.

NOTE : 1.1.4

An isoform lattice is always regular.

NOTATION : 1.1.5

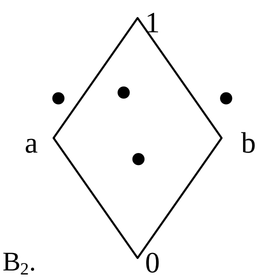
For a lattice L, we denote by ω_L and i_L the smallest and the largest congruence on L, respectively.

C_n will denote the n element chain.

B_n will denote the Boolean algebra with 2^n elements. For a bounded lattice A with bounds 0 and 1, A^- will denote the lattice A-{0, 1}

EXAMPLE : 1.1.6

Consider the Boolean algebra B_2, with 4 elements.



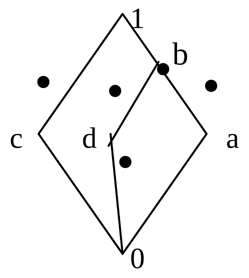
Its congruence lattice is also B_2.

It has four congruences, namely, the null congruence ω, the all congruence i and two non-trivial congruences θ_1 and θ_2. θ_1 has two congruence classes { {0,a}, {1,b} } and θ_2 has two congruence classes { {0,b} } , {a,1} }.

[0,a] and [b,1] are isomorphic and [0,b] and [a,1] are isomorphic. So, θ_1 and θ_2 are isoform congruences. Trivially ω and i are isoform congruences. Hence B_2 is a isoform lattice.

NOTE : 1.1.7

Every lattice need not be an isoform lattice. For example, the lattice N_6, given below is not isoform.



This lattice has exactly one non-trivial congruence θ and θ has exactly two congruence classes {0,a,b,d} and {c,1}. These two congruence classes are not isomorphic.

Hence θ is not an isoform congruence.

$\therefore$ L is not an isoform lattice.

DEFINITION : 1.1.8

Let L be a lattice and [a,b] an interval of L. If θ is a congruence on L, we call θ discrete on [a,b] or [a,b] is θ-discrete, if θ and ω agree on [a,b].

That is $\theta|_{[a,b]} = \omega_{[a,b]}$.

That is $\theta|_{[a,b]} = \{(a,a),(b,b)\}$.

DEFINITION : 1.1.9

Let L be a finite lattice. We call L discrete-transitive, if for any congruence Φ of L and for a < b < c in L, whenever Φ is discrete on [a,b] and on [b,c] then Φ is discrete on [a,c] .

DEFINITION : 1.1.10

Let $P=(P,\leq_P)$ be a finite poset. Then the partial ordering $\leq_P$ on P is the reflexive - transitive extension of $\prec_P$, the covering relation on $(P, \leq_P)$. That is ReflTr $(\prec_P) = \leq_P$.

Let H be a subset of $\prec_P$. Take the reflexive-transitive extension ReflTr(H) of H. Then (P,ReflTr(H)) is also a poset. This is called as a pruning of P. The diagram of (P, ReflTr(H)) can be obtained from the diagram of $(P,\leq)$ by cutting out some edges but not deleting any elements.

1.2. A Lattice Construction

DEFINITION : 1.2.1

Let A be a nontrivial finite bounded lattice with bounds 0 and 1 and $|A|>2$.

Let B be a nontrivial finite lattice with a discrete transitive congruence θ.

For $u \in AxB$, we use the notation $u=(u_A,u_B)$ where $u_A \in A$ and $u_B \in B$. We shall denote by $\leq_X$, $\prec_X$, V_X and Λ_X, the partial ordering, the covering relation, the join and the meet on AxB respectively.

Let $B_*=\{0\}$ x B, $B^*=\{1\}$ x B, and for $b \in B$, let $A_b=Ax\{b\}$.

We define the set Prune(A,B,θ) by

Prune(A,B,θ)=$\{((a,b_1), (a,b_2)) / a \in A^{-}, b_1 \prec b_2$ in B and $b_1 \equiv b_2(\theta)\}$.

Then Prune (A, B, θ) is a subset of $\prec_X$.

Define H = $\prec_X$ - Prune (A,B,θ).

Consider the reflexive, transitive closure of H.

Then ReflTr(H) is a partial order on AxB.

Define N(A,B,θ)=(AxB, ReflTr(H))

We shall denote the partial ordering ReflTr(H) on AxB by $\leq_{N(A,B,\theta)}$ or simply by $\leq_N$.

NOTE : 1.2.2

If $\theta = \omega$, then $N(A,B,\theta)$ is the direct product AxB.

PROPOSITION : 1.2.3

Let $u,v \in AxB$ and $u \leq_X v$. Then $u \leq_N v$ if, and only if,

(i) $u_A, v_A \in A^-$ and $[u_B, v_B]$ is θ - discrete (or)

(ii) u_A or $v_A \notin A^-$

Proof :-

Let $\leq_F$ denote the binary relation on $N(A,B,\theta)$ defined by $u \leq_F v$ if, and only if, (i) or (ii) holds. We claim that $\leq_F$ is a partial order.

Trivially $\leq_F$ is reflexive and anti-symmetric.

To prove $\leq_F$ is transitive.

Let $u \leq_F v$ and $v \leq_F w$

Then $u \leq_X w$. We have to distinguish some cases.

Case : 1

Both $u \leq_F v$ and $v \leq_F w$ hold by (i)

Then $u_A, v_A \in A^-$ and $v_A, w_A \in A^-$ imply $u_A, w_A \in A^-$.

$[u_B, v_B]$ is θ-discrete, $[v_B, w_B]$ is θ-discrete and θ is discrete transitive imply $[u_B, w_B]$ is θ-discrete.

Hence by (i), $u \leq_F w$.

Case : 2

$u \leq_F v$ holds by (i) and $v \leq_F w$ holds by (ii)

$\therefore u_A, v_A \in A^-$ and $[u_B, v_B]$ is θ-discrete and v_A or $w_A \notin A^-$.

$u_A, v_A \in A^-$ forces $w_A \notin A^-$.

Hence by (ii) $u \leq_F w$.

Case : 3

$u \leq_F v$ holds by (ii) and $v \leq_F w$ holds by (i)

$u \leq_F v$ holds by (ii) implies either u_A or $v_A \notin A^-$

$v \leq_F w$ holds by (i) implies $v_A, w_A, \in A^-$ and $[v_B, w_B]$ is θ-discrete.

v_A, w_A, $\in A^-$ and u_A or $v_A \notin A^-$ forces $u_A \notin A^-$.

$u_A \notin A^-$ implies $u \leq_F w$ by (ii).

Case 4 :

Both $u \leq_F v$ and $v \leq_F w$ holds by (ii).

That is, u_A or $v_A \notin A^-$ and v_A or $w_A \notin A^-$.

If u_A or $w_A \notin A^-$, then by (ii) $u \leq_F w$ holds.

Suppose that u_A, $w_A \in A^-$. Then $v_A \notin A^-$.

$\therefore v_A=0$ or 1 and $u_A \leq_X v_A \leq_X w_A$

If $v_A=0$, then $u_A=0$, contradicting that $u_A \in A^-$.

If $v_A=1$, the $w_A=1$, contradicting that $w_A \in A^-$.

These two contradictions prove that $v_A \notin A^-$ is impossible.

$\therefore$ Either u_A or $w_A \notin A^-$

Hence $u \leq_F w$ holds.

$\therefore \leq_F$ is a partial order.

If u □v, then $u \leq_F v$ if, and only if, $u \leq_N v$

Hence $\leq_F = \leq_N$.

Hence the proposition.

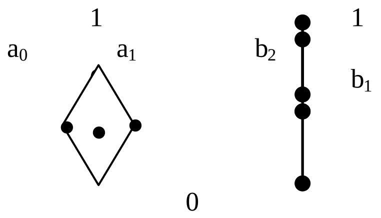
NOTE : 1.2.4

$u \leq_X v$ and $u \nleq_N v$ if, and only if,

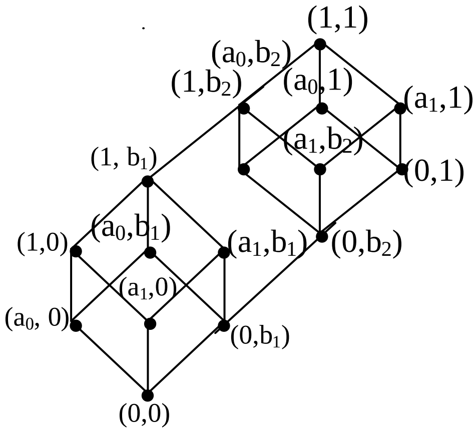
u_A, $v_A \in A^-$ and $[u_B, v_B]$ is not θ - discrete.

EXAMPLE : 1.2.5

Let $A = B_2$ and $B = C_4$ and $\theta = \theta_{(b_1, b_2)}$



Then θ is discrete transitive. The lattice N (A, B, θ) is given below.



LEMMA : 1.2.6

N (A,B,θ) is a lattice under the partial ordering $\leq_N$. The meet and join in N (A, B, θ) can be computed using the formulae.

$$(*)\quad u\Lambda_N v = \begin{cases} u\Lambda_X v & \text{if } u\Lambda_X v \leq_N u \text{ and } u\Lambda_X v \leq_N v \\ (0, u_B\Lambda v_B), & \text{otherwise} \end{cases}$$

and

$$(**)\quad uV_N\nu = \begin{cases} uV_X\nu & \text{if } u \leq_N uV_X\nu \text{ and } \nu \leq_N uV_X\nu \\ (1, u_BVv_B) & \text{otherwise.} \end{cases}$$

Proof :-

Let u, v $\in$A x B and let t be a lower bound of u and ν in N (A, B, θ).

Case 1 :

$u\Lambda_X\nu$ is not a lower bound of both u and ν in N(A,B, θ).

Suppose $u\Lambda_X \nu \Leftarrow u$.

Then by (1.2.4), $u_A\Lambda v_A$, $u_A \in A^-$ and $[u_B\Lambda v_B, u_B]$ is not θ-discrete.

It follows that $[t, u_B]$ is not θ-discrete.

So, $t \leq_N u$ implies by (ii) of proposition (3.2.3), $t_A \notin A^-$.

We cannot have $t_A=1$, for it will imply $u_A=1$, contradicting $u_A \in A^-$.

Therefore $t_A = 0$.

Therefore $t \leq (0, u_B\Lambda v_B)$.

Hence $u \Lambda_N v = (0, u_B \Lambda v_B)$.

Similarly, if $u\Lambda_X v \Leftarrow \nu$, then also we can prove $u \Lambda_N v = (0, u_B\Lambda v_B)$

Thus in this case $u \Lambda_N v = (0, u_B \Lambda v_B)$

Case 2 :

$u \Lambda_X v$ is a lower bound of both u and v in N(A,B,θ).

If $t \Leftarrow_N u \Lambda_x v$, then by (3.2.4.) t_A, $u_A \Lambda v_A \in A^-$ and $[t, u\Lambda_x v]$ is not θ-discrete.

Since $u_A \Lambda v_A \in A^-$, it follows that $u_A \in A^-$ or $v_A \in A^-$ or both $u_A \in A^-$ and $\nu_A \in A^-$.

Suppose $u_A \in A^-$.

Since $t \leq_N u$ and t_A, $u_A \in A^-$, we conclude that $[t,u]$ is θ-discrete by (i) of

proposition 1.2.3.
This contradicts the fact that [t, $u\Lambda_X v$] is not θ-discrete.
Hence t $\leq_N u \Lambda_X v$.
Thus any lower bound of both u and v is $\leq_N u \Lambda_X v$.
Hence $u \Lambda_N v = u \Lambda_X v$.
Similarly if $v_A \in A^-$, then also $u\Lambda_N v = u \Lambda_X v$.
Thus in case 2, $u\Lambda_N v = u\Lambda_X v$.
Hence (*) holds in N(A, B, θ).
By duality (* *) holds in N(A, B, θ).
Hence N(A, B, θ) is a lattice.

1.3.The Congruences On N (A, B, θ)

DEFINITION : 1.3.1

Let A be a bounded lattice. A congruence Φ of A is said to separate 0 if [0] Φ = {0}.
That is $x \equiv 0(\Phi)$ implies that x = 0.

DEFINITION : 1.3.2

Let A be a bounded lattice. A congruence Φ of A is said to separate 1 if [1] (Φ) = {1}.
That is $x \equiv 1 (\Phi)$ implies that x = 1.

DEFINITION : 1.3.3

Let A be a bounded lattice. Then A is said to be non-separating if neither 0 nor 1 is separated by any congruence $\Phi \neq \omega$ of A.

NOTE : 1.3.4

In this section, we assume that A is a non-separating finite lattice with more than two elements. B is a finite lattice with more than one element and $\theta > \omega$ is a discrete-transitive congruence on B.

LEMMA : 1.3.5

Let ψ be a congruence relation on N(A, B, θ). Define ψ_* and ψ^* as the restriction of ψ to B_* and B^* respectively. Since B_* and B^* are isomorphic to B, we can view ψ_* and ψ^* as congruences on B.
Then $\psi_* = \psi^*$.

Proof : -

Let $b_0 \equiv b_1 (\psi_*)$.
Then $(0, b_0) \equiv (0, b_1) (\psi)$
Joining both sides with (1,0) we get
$(0, b_0) \vee (1,0) \equiv (0, b_1) \vee (1, 0) (\psi)$
That is $(1, b_0) \equiv (1, b_1) (\psi)$
That is $b_0 \equiv b_1 (\psi^*)$
Thus $b_0 \equiv b_1 (\psi_*)$ implies $b_0 \equiv b_1 (\psi^*)$.
Hence $\psi_* \leq \psi^*$.
Similarly, we can prove that $x \equiv y (\psi^*)$ implies $x \equiv y (\psi_*)$.
Hence $\psi^* \leq \psi_*$
Thus we get $\psi^* = \psi_*$

NOTE : 1.3.6

Let ψ be a congruence relation on N(A,B,θ). For any b∈B, A_b is isomorphic to A. Define ψ_b as the restriction of ψ to A_b. Then ψ_b is a congruence on A.

LEMMA : 1.3.7

Let ψ be a congruence relation on N(A,B,θ). The congruences $\psi_* = \psi^*$ of B and the family of congruences $\Gamma\psi = \{\psi_{\mathbf{b}} / b \in B\}$ of A describe the congruence ψ of N(A,B,θ).

Proof :-

We know that in a finite lattice a congruence is completely determined by the set of prime intervals it collapses.

Every prime interval of N (A,B,θ) is in one of the sublattices B_* , B^* or A_b for some $b \in B$, or is perspective to a prime interval of B_*.

Hence ψ is determined by $\psi_* = \psi^*$ or by $\{\psi_{\mathbf{b}} / b \in B\}$.
Hence the lemma.

LEMMA : 1.3.8

Let ψ be a congruence relation on N(A,B, θ). For any b$\in$B, let ψ_b be the restriction of ψ to A_b. The family of congruences $\lceil \psi = \{\psi_b / b\in B\}$ is either $\{ \omega_A / b \in B\}$ or $\{i_A / b\in B\}$.

Proof :-

Let us assume that $x < y \in A_b$ for some b$\in$B and $x \equiv y\ (\psi)$.

Since A is non-separating, we can assume that x (0, x_B) the zero of A_b.

$(0, x_B) \equiv y\ (\psi) \Rightarrow (0, x_B)\ V\ (0,1) \equiv (0,1)\ V\ y\ (\psi)$

$\Rightarrow (0,1) \equiv_b (0,1)\ V\ y\ (\psi)$

But $(0,\overset{b}{1}) < (0,1)\ V\ y$.

So we can assume that x < y in A_1.

As A is non-separating, x, y $\in A_1$, $x \equiv y\ (\psi)$, we can assume that $y_A = 1$ (ie) $y = (1, y_B)$.

If $x_A = 0$, then $\psi_1 = i_A$.

The congruence $x \equiv y\ (\psi)$ implies $(1, b)\ \Lambda\ x \equiv (1,b)\ \Lambda\ y\ (\psi)$

That is $(0,b) \equiv (1,b)\ (\psi)$

Hence $\psi_b = i_A$.

Suppose $x_A \neq 0$.

$\theta > \omega$ by assumption. So, the interval [0,1] of B is not θ-discrete.

$x \equiv y\ (\psi)$ implies $x\ \Lambda\ (1,0) \equiv y\ \Lambda\ (1,0)\ (\psi)$.

That is $(1,0) \equiv (0,0)\ (\psi)$.

That is $\psi_0 = i_A$.

$(1,0) \equiv (0,0)\ (\psi)$ implies $(1,0)\ V\ (0,b) \equiv (0,0)\ V\ (0,b)\ (\psi)$

That is $(1,b) \equiv (0,b)\ (\psi)$

Hence $\psi_b = i_A$.

LEMMA : 1.3.9

Let A be a finite non-separating lattice with more than two elements. Let B be a finite lattice with more than one element and $\theta > \omega$ be a discrete -transitive congruence on B. Consider N(A,B,θ). For every congruence Φ of B, there exists a unique minimal congruence N(Φ) of N(A,B,θ) satisfying $N(\Phi)_* = N(\Phi)^* = \Phi$. The congruence N($\Phi$) of N(A,B,$\theta$) can be described as follows :

$$N(\Phi) = \begin{cases} \omega_A \times \Phi, & \text{if } \Phi\ \Lambda\theta = \omega \\ i_A \times \Phi, & \text{if } \Phi\ \Lambda\theta > \omega \end{cases}$$

Proof :-

Case1: Let us assume that $\Phi\ \Lambda\ \theta = \omega$

Let $\psi = \omega_A \times \Phi$

As ω_A and Φ are equivalence relations, ψ is also an equivalence relation.

Let $x \equiv y\ (\psi)$. Then $x_A \equiv y_A\ (\omega_A)$ and $x_B \equiv y_B\ (\Phi)$

As ω_A and Φ are congruence relations we have

$x_A \Lambda y_A \equiv x_A V y_A\ (\omega_A)$ and $x_B \Lambda y_B \equiv x_B V y_B\ (\Phi)$

Hence we have $x\ \Lambda y \equiv xVy\ (\psi)$.

To prove ψ is a congruence relation, it is sufficient to verify that

(***) For x, y $\in$ N(A, B,θ) with x < y and for t$\in$N(A,B,θ)

if $x \equiv y\ (\psi)$ then $x\Lambda t \equiv y\Lambda t\ (\psi)$ and $xVt \equiv yVt\ (\psi)$.

Let $x \equiv y\ (\psi)$. Then $x_A \equiv y_A\ (\omega_A)$ and $x_B \equiv y_B\ (\Phi)$.

$x_A \equiv y_A\ (\omega_A)$ implies $x_A = y_A$.

Thus we have $x_A = y_A \rightarrow (1)$ and $x_B \equiv y_B\ (\Phi) \rightarrow (2)$

We have to prove $x\Lambda t \equiv y\Lambda t\ (\Phi)$.

That is $(x\Lambda t)_A = (y\Lambda t)_A \rightarrow (3)$ and

$(x\Lambda t)_B \equiv (y\Lambda t)_B\ (\Phi) \rightarrow (4)$

By lemma (1.2.6) (*), $(x\Lambda t)_B = x_B\Lambda t_B$ and $(y\Lambda t)_B = y_B\Lambda t_B$. .

Hence (4) can be written as $x_B\Lambda t_B \equiv y_B\Lambda t_B(\Phi)$.

By (2) $x_B \equiv y_B\ (\Phi)$ and Φ is a congruence on B.

implies $x_B\Lambda t_B \equiv y_B\Lambda t_B\ (\Phi)$.

Hence (4) holds.

It remains to prove that $(x\Lambda t)_A = (y\Lambda t)_A \rightarrow (3)$

By assumption, $\Phi\ \Lambda\ \theta = \omega$

Hence $x_B\Lambda t_B \equiv y_B\Lambda t_B\ (\Phi)$ can be written as

$[x_B \Lambda t_B, y_B \Lambda t_B]$ is θ-discrete → (5)
$(y\Lambda t)_A = y_A \Lambda t_A$ or $(y\Lambda t)_A = 0$ (by lemma 1.2.6 (*))
If $(y\Lambda t)_A = 0$, then $(x\Lambda t)_A \leq (y\Lambda t)_A = 0$ implies $(x\Lambda t)_A = 0$.
Hence $(x\Lambda t)_A = (y\Lambda t)_A = 0$
Hence (3).
Suppose $(y\Lambda t)_A = y_A \Lambda t_A$. That is $y\Lambda t = y\Lambda_X t$.
Then we prove that $x\Lambda t = x\Lambda_X t$ and from this (3) follows $y\Lambda t = y\Lambda_X t$.
By lemma (1.2.6) (*), this is equivalent to
$y\Lambda t \leq_N y$ and $y\Lambda t \leq_N t$, which can be rewritten as follows :

One of the following conditions holds :
$y_A \Lambda t_A = 0$ or 1 → (1a)
$y_A = 0$ or 1 → (1b)
$[y_B \Lambda t_B, y_B]$ is θ-discrete → (1c)

and one of the following conditions holds :
$y_A \Lambda t_A = 0$ or 1 → (2a)
$t_A = 0$ or 1 → (2b)
$[y_B \Lambda t_B, t_B]$ is θ-discrete -->(2c)

We have to prove that $x\Lambda t = x\Lambda_X t$
By, lemma (1.2.6) (*), this is equivalent to $x\Lambda t \leq_N x$ and $x \Lambda t \leq_N t$
That is one of the following conditions holds :
$x_A \Lambda t_A = 0$ or 1 → (3a)
$x_A = 0$ or 1 → (3b)
$[x_B \Lambda t_B, x_B]$ is θ-discrete →(3c)
and one of the following conditions holds :
$x_A \Lambda t_A = 0$ or 1 → (4a)
$t_A = 0$ or 1 →(4b)
$[x_B \Lambda t_B, t_B]$ is θ-discrete→(4c)

Assume (1a, 1b, 1c)

Since $x_A = y_A$, $y_A \Lambda t_A = 0$ or 1 implies
$x_A \Lambda t_A = 0$ or 1
Hence (1a) implies (3a).
$x_A = y_A$, $y_A = 0$ or 1 implies $x_A = 0$ or 1.
Hence (1b) implies (3b).
By (1c) we have $[y_B \Lambda t_B, y_B]$ is θ-discrete.
By(5) $[x_B \Lambda t_B, y_B \Lambda t_B]$ is θ-discrete.
Since θ is discrete - transitive, we conclude that $[x_B \Lambda t_B, y_B]$ is θ-discrete.
Hence $[x_B \Lambda t_B, x_B]$ is θ-discrete ($x_B < y_B$).
Hence (1c) implies (3c).
Thus (1a, 1b, 1c) imply (3a, 3b, 3c).

Next assume that (2a, 2b, 2c) hold.

Since $x_A = y_A$, $y_A \Lambda t_A = 0$ or 1 implies $x_A \Lambda t_A = 0$ or 1
That is (2a) implies (4a).
By (2b) $t_A = 0$ or 1, which is the same as (4b).
Finally (2c) gives $[y_B \Lambda t_B, t_B]$ is θ-discrete.
By (5) $[x_B \Lambda t_B, y_B \Lambda t_B]$ is θ-discrete.
Since θ is discrete-transitive, we conclude that
$[x_B \Lambda t_B, t_B]$ is θ-discrete.
Hence (4c).
Thus (2a, 2b, 2c) imply (4a, 4b, 4c).
Thus $y \Lambda t = y \Lambda_x t$ implies $x \Lambda t = x \Lambda_X t$.

Case : 2

Let us assume that $\Phi \Lambda \theta > \omega$
Define $\psi = i_A X \Phi$
Then ψ is a congruence relation on N(A,B,θ).
Moreover $\psi_* = \psi^* = \Phi$ and $\psi_b = i_A$ for all $b \in B$.
Next we claim that N(Φ) is a minimal congruence of N(A,B,θ) satisfying $N(\Phi)_* = N(\Phi)^* = \Phi$.
Let Σ be a congruence of N(A, B, θ) satisfying $\Sigma_* = \Sigma^* = \Phi$

Since $\Phi \wedge \theta > \omega$, we can choose in B the elements $b_1 \square\, b_2$ such that
$b_1 \equiv b_2\ (\Phi \wedge \theta)$.
From $\Sigma_* = \Phi$, it follows that $b_1 \equiv b_2\ (\Sigma)$ also holds.
By assumption, A has more than two elements, so we can
choose $a \in A^-$.
$(a,b_1) \vee (0,b_2) = (1, b_1 \vee b_2) = (1,b_2)$.
Since $b_1 \equiv b_2\ (\Sigma_*)$, it follows that $(0,b_1) \equiv (0,b_2)\ (\Sigma)$.
Joining both sides with (a,b_1) we get
$(a,b_1) \vee (0,b_1) \equiv (a,b_1) \vee (0,b_2)(\Sigma)$
$(a,b_1) \equiv (1,b_2)(\Sigma)$
$\therefore (a,b_1) \equiv (l,b_1)(\Sigma)$.
Thus we get $\Sigma_{b_1} > \omega_A$
By lemma (1.3.8), we get $\Sigma_{b_1} = i_{A}$ 1
$\therefore\ \Sigma_b = i_A$ for all $b \in B$. 1 b 1
Hence $\Sigma \geq \psi$. b
$\therefore \psi = N(\Phi)$ is the smallest congruence of $N(A,B,\theta)$ satisfying
$\psi_* = \psi^* = \Phi$.

By the lemma (1.3.9), we conclude that to every congruence Φ of B, we can associate a congruence $N(\Phi)$ of $N(A,B,\theta)$. In the next lemma, we see some properties of the map which associates Φ to $N(\Phi)$.

LEMMA : 1.3.10

Let A be a finite non-separating lattice with more than two elements. Let B be a finite lattice with more than one element and $\theta > \omega$ is a discrete-transitive congruence on B. consider $N(A,B,\theta)$. Define a map
$N : \mathrm{Con}B \rightarrow \mathrm{Con}N(A,B,\theta)$ by $N(\Phi)=N(\Phi,A,B,\theta)$ which is denoted by $N(\Phi)$.
Then

(i) N is an order preserving, one-to-one map of ConB into $\mathrm{Con}N(A,B,\theta)$.
(ii) The map N is an order preserving, one-to-one map of the join irreducible elements of ConB into join-irreducible elements of $\mathrm{Con}N(A,B,\theta)$.
(iii) The lattice $\mathrm{Con}N(A,B,\theta)$ has exactly one join-irreducible element that is not in the image of N. $\Sigma = \theta\ (\ (0,0),(1,0)\ )$, Σ is a minimal join-irreducible element of $\mathrm{Con}N(A,B,\theta)$.
(iv) For a minimal join-irreducible congruence Φ of B, we have $\Sigma < N(\Phi)$ iff, $\Phi \leq \theta$.

Proof :-

By lemma 1.3.9., if $\Phi \in \mathrm{Con}B$, then $N(\Phi)$ is the unique minimal congruence of $N(A,B,\theta)$ such that $N(\Phi)_* = N(\Phi)^* = \Phi$.
Hence if $\Phi_1, \Phi_2 \in \mathrm{Con}B$, and if $N(\Phi_1) = N(\Phi_2)$ then $\Phi_1 = \Phi_2$.
That is N is one-one.
If $\Phi_1 \leq \Phi_2$ then $N(\Phi_1) \leq N(\Phi_2)$.
Hence N is an order preserving one-to-one map of ConB into $\mathrm{Con}N(A,B,\theta)$.
A join-irreducible congruence of a finite lattice is one that is generated by a covering pair of elements.
If Φ is a join-irreducible congruence, then $\Phi = \theta\ (b_1,b_2)$ with
$b_1 \square\, b_2$ in B.
Then $N(\Phi) = \theta\ (\ (0,b_1), (0,b_2)\ )$ and $(0,b_1) \square\ (0, b_2)$ in $N(A,B,\theta)$. So, the join-irreducible congruences of B are mapped by N into join-irreducible congruences of $N(A,B,\theta)$.
Also N is order preserving and one-one.
Hence (ii).
Any prime interval of $N(A,B,\theta)$ is in one of the sublattices B_*, B^*, or A_b, for some $b \in B$, or is perspective to a prime interval of B_*.

The prime intervals in B_* and B^* generate the join-irreducible congruences of the form $N(\Phi)$, where Φ is a join-irreducible congruence of B.
The remaining prime intervals all generate the same join-irreducible congruence Σ, by lemma 1.3.8.
Thus the lattice $\mathrm{Con}N(A,B,\theta)$ has exactly one join-irreducible element that is not in the image of N.
Hence (iii).

$\Sigma < N(\Phi)$ holds if, and only if, $\Phi \wedge \theta > \omega$

If $\Phi \wedge \theta < \Phi$, then there is a join-irreducible congruence of B, namely $\theta \wedge \Phi$ properly below Φ, contrary to our assumption.

Therefore $\theta \wedge \Phi = \Phi$.

That is $\Phi \leq \theta$.

Thus $\Sigma < N(\Phi)$ if, and only if, $\Phi \leq \theta$.

Hence (iv) .

Hence the lemma.

REMARK : 1.3.11

Let D be a finite distributive lattice.

Let J(D) denote the poset of join-irreducible elements of D.

For a minimal join-irreducible element p of D, let Cov(p) denote the covers of p in J(D)

That is Cov(p) = { $q \in J(D)$ / p □ q}.

Let D' denote the join-subsemilattice of D generated by J(D) – {p}.

Then D' is a finite distributive lattice with J(D') = J (D) – {p}. The set Cov(p) is an antichain of J(D').

Conversely, given a finite distributive lattice D' and an antichain $C \neq \phi$ of J(D'), we can form the poset $J(D') \cup \{p\}$ where $p \notin J(D')$.

We can extend the partial ordering of J(D') to $J(D') \cup \{p\}$ by defining $p < q$ for all $q \in C$.

More precisely, we define $p < r$ for every $r \in J(D')$ for which there exists a $q \in C$ satisfying $q \leq r$.

The poset $J(D') \cup \{p\}$ determines a distributive lattice D.

In D, Cov(p) = C.

We call D', the distributive lattice obtained from D by deleting the minimal join-irreducible element p and we call D, the distributive lattice obtained from D' by adding a minimal join-irreducible element under C.

Next we summarize the properties we have learned about the congruence lattice of $N(A,B,\theta)$.

THEOREM : 1.3.12

Let A be a finite non-separating lattice with more than two elements. Let B be a finite lattice with more than one element, and let $\theta > \omega$ be a discrete-transitive congruence on B.

Let $\theta = \Sigma_1 \vee \Sigma_2 \vee \ldots\vee \Sigma_n$ be an irredundant representation of θ as a join of join-irreducible elements and let $C=\{\Sigma_1, \Sigma_2 \ldots\ldots \Sigma_n\}$. Let Σ be the join-irreducible congruence of $N(A,B,\theta)$ define by $\Sigma = \theta((0,0), (1,0))$.

Then we can obtain, upto isomorphism, the congruence lattice of $N(A,B,\theta)$ by adjoining to the congruence lattice of B, a minimal join-irreducible element under C.

Equivalently, we can obtain, upto isomorphism, the congruence lattice of B by deleting the minimal join-irreducible element Σ of $\mathrm{Con}N(A,B,\theta)$.

LEMMA : 1.3.13

Let A be a finite non-separating lattice with more than two elements. Let B be a finite lattice with more than one element and let $\theta > \omega$ be a discrete-transitive congruence on B. Consider $N(A,B,\theta)$.

If Φ is a discrete-transitive congruence of B, then the congruence $N(\Phi)$ of $N(A,B,\theta)$ is also a discrete-transitive congruence.

Proof :-

Case (1)

Let us assume that $\Phi \wedge \theta = \omega$

Then $N(\Phi) = \omega_A \times \Phi$ by lemma (3.3.9).

For elements $a < b \in N(\Phi)$, $a \equiv b\ (N(\Phi))$ if, and only if, $a_A = b_A$ and $a_B \equiv b_B\ (\Phi)$.

Therefore, an interval [u,v] of $N(A,B,\theta)$ is $N(\Phi)$-discrete if, and only if, the interval $[u_B,v_B]$ of B is Φ-discrete.

Therefore, if Φ is discrete-transitive in B, then $N(\Phi)$ is discrete- transitive in $N(A,B,\theta)$.

Case (2)

Let us assume that $\Phi \wedge \theta > \omega$.

Then $N(\Phi) = i_A \times \Phi$ by lemma (3.3.9).

Then an interval [u,v] of $N(A,B,\theta)$ is $N(\Phi)$ discrete if, and only if, $u_A= v_A$ and the interval $[u_B,v_B]$ of B is Φ-discrete.

As Φ is discrete-transitive, it follows that $N(\Phi)$ is also discrete-transitive.

Hence the lemma.

COROLLARY : 1.3.14

Let A be a finite non-separating lattice with more than two elements. Let B be a finite lattice with more than one element and let $\theta > \omega$ be a discrete-transitive congruence on B. If all congruences of B are discrete-transitive, then all congruences of N(A,B,θ) are discrete-transitive.

Proof :-

First, we observe that the congruence $\Sigma = \theta$ ((0,0),(1,0)) is discrete – transitive.

Any congruence of N(A,B,θ) is of the form ΣVN(Φ) where Φ is a congruence of B. We know that the join of two discrete-transitive congruence is discrete-transitive.

Since Φ is discrete-transitive, N(Φ) is also discrete-transitive.

Hence ΣVN (Φ) is discrete-transitive.

Thus all congruences of N(A,B,θ) are discrete-transitive if all congruences of B are discrete-transitive.

Hence the result.

1.4. The Main Theorem

In this section we prove the theorem given below :

THEOREM : 1.4.1

Every finite distributive lattice D can be represented as the congruence lattice of a finite lattice L with the following properties :

(i) L is isoform.
(ii) For every congruence θ of L, the congruence classes of θ are projective intervals.
(iii) L is a finite pruned Boolean lattice.
(iv) L is discrete-transitive.

Proof:-

Let D be a finite distributive lattice.

We have to construct a lattice L satisfying the conditions of the above theorem.

If D is the one-element lattice, then let L be the one-element lattice.

If D has more than one element, then J(D) $\neq \phi$.

We prove the result using induction on n = $|$J(D)$|$.

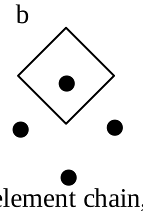
If $|$J(D)$|$ = n = 1, then let L = C_2.

If $|$J(D)$|$ = 2, then either J(D) is unordered or J(D) is a two-element chain.

If J(D) is unordered, choose L = C_2^2

1

(ie) L = B_2 = a b



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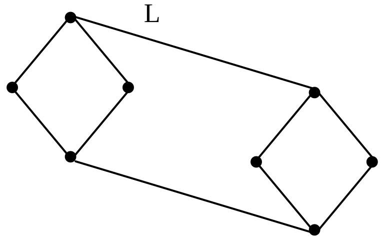
If J(D) is the two element chain, then choose L as the lattice given below.

L



Then L satisfies the conditions of the above theorem.

By induction assumption assume that the result is true when $|$J(D)$|$ < n.

Now, we prove the result when $|$J(D)$|$ = n > 2.

If J(D) is an antichain, the theorem follows, by choosing the lattice L as the Boolean lattice with n atoms.

If J(D) is not an antichain, choose a minimal but not maximal join-irreducible element p of D.

Let D' be the distributive lattice join-generated by J(D) – {p}.

Then $|J(D')| = n - 1$.

Then by induction assumption, there is a lattice B and a lattice isomorphism

$\alpha : D' \rightarrow$ Con B satisfying the conditions of the above theorem.

Since p is not a maximal element of J(D), it follows that Cov(p) $\neq \phi$.

Let $q \in$ Cov(p).

Since $q \in D'$, under the isomorphism α is mapped to a congruence θ_q of B.

Define $\theta = V\{ \theta q \mid q \in Cov(p) \}$

Let $L = N(B_2, B, \theta)$

By theorem 1.3.12., Con L $\cong$ D.

Now, we have to prove L satisfies conditions (i) to (iv) of the theorem.

Let ψ be a congruence of L.

Then ψ is one of the following forms by lemma 1.3.9, lemma 1.3.10 and theorem 1.3.11.

Form : 1

$\psi = N(\Phi) = \omega_A \times \Phi$, where Φ is a congruence of B satisfying $\theta \Lambda \Phi = \omega$.

Form : 2

$\psi = N(\Phi) = i_A \times \Phi$, where Φ is a congruence of B satisfying $\Phi \Lambda \theta > \omega$.

Form : 3

$\psi = N(\Phi) V \Sigma$, where Φ is a congruence of B.

Form 1 : $\psi = N(\Phi) = \omega_A \times \Phi$.

Then the congruence classes of $N(\Phi)$ are described as follows.

Let [u, v] be a congruence class of Φ in B.

Then the congruence classes of ψ in L are exactly the intervals of the form [(a,u), (a,v)] for any $a \in A$.

The interval [u,v] of B is isomorphic to the interval [(a,u), (a,v)] of L by a $\notin A^-$ and by lemma 1.2.6.

If [u,v] and [u',v'] are any two congruence classes of Φ in B, then [u,v] and [u',v'] are isomorphic intervals and they are projective by induction hypothesis.

Then [(a,u),(a,v)] and [(a',u'),(a',v')] are isomorphic for any a, $a' \in A$.

We have to prove that [(a,u),(a,v)] and [(a',u'),(a',v')] are projective.

[(a,u),(a,v)] is perspective to [(0,u),(0,v)] and [(a',u'),(a',v')] is perspective to [(0,u'),(0,v')].

Therefore, to prove [(a,u),(a,v)] and [(a',u'),(a',v')] are projective, it is enough to prove [(0,u),(0,v)] and [(0,u'),(0,v')] are projective.

By induction assumption, [u,v] and [u',v'] are projective.

A trivial induction shows that it is sufficient to verify that if [u,v] and [u',v'] are perspective, then so are [(0,u),(0,v)] and [(0,u'),(0,v')].

By duality it is sufficient to compute this for up perspectives.

So, let $v \Lambda u' = u$ and $v V u' = v'$.

Then $(0,v) \Lambda (0,u') = (0,u)$ and $(0,v) V (0,u') = (0,v')$.

This completes the proof in this case.

If ψ is of form 2 or form 3, then the congruence classes of ψ are described in lemmas 1.3.9. and **1.3.10. as follows:**

Let [u, v] be a congruence class of Φ in B.

Then the congruence classes of ψ in L are exactly the intervals of L of the form [(0,u), (1,v)].

[(0,u), (1,v)] is isomorphic to $N(B_2, [u,v], i_{[u,v]})$.

So, if the intervals [u,v] and [u',v'] of B are isomorphic, so are the intervals [(0,u),(1,v)] and [(0,u'),(1,v')] of L.

We have to prove that any two congruence classes of ψ are projective intervals.

Let [u,v] and [u',v'] be any two congruence classes of Φ in B.

Then [(0,u),(1,v)]and[(0,u'),(1,v')] are the corresponding ψ classes in L.

If [u,v] is up perspective to [u',v'] then $v V u' = v'$ and $v \Lambda u' = u$.

$\therefore$ (1,v) V (0,u') = (1,v V u') = (1,v') and

$(1,v) \Lambda (0,u') = (0, v \Lambda u') = (0,u)$

$\therefore$ [(0,u),(1,v)] and [(0,u'),(1,v')] are up perspective.

Similarly, if [u,v] is down perspective to [u',v'], then we get [(0,u),(1,v)] and [(0,u'),(1,v')] are down perspective (by duality).

If [u,v] and [u',v'] are projective, then [(0,u),(1,v)] and [(0,u'),(1,v')] are projective by induction.

This completes the proof of conditions (i) and (ii).

Condition (iii) is obvious.

By hypothesis, B is a pruned Boolean lattice.

Of course, B_2 is a Boolean lattice.

So, L is a pruned Boolean lattice.

Finally by corollary 1.3.14, the congruence's of L are discrete-transitive.

Hence the theorem.

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