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Some Common Fixed Point Results for Expansive Mappings in a Cone Metric Space

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***Abstract*:** *The purpose of this work is to extend and generalize some common fixed point theorems for Expansive type mappings in complete cone metric spaces. We are attempting to generalize the several well-known recent results.*
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Key word: *Complete cone metric space, common fixed point, expansive type mapping.*

I. Introduction

Very recently, Huang and Zhang [3] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space. They prove some fixed point Theorems for contractive mappings using normality of the cone. The results in [3] were generalized by Sh. Rezapour and Hamlbarani [4] omitted the assumption of normality on the cone, which is a milestone in cone metric space.

In this manuscript, the known results [14] are extended to cone metric spaces where the existence of common fixed points for expansive type mappings on cone metric spaces is investigated.

II. Preliminary Notes

Definition 2.1[3] : Let E be a real Banach space and P, a subset of E. Then P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$;
(ii) $a, b \in R$, $a,b \geq 0$ $x,y \in \boldsymbol{P} \Rightarrow ax+by \in \boldsymbol{P}$;
(iii) $x \in$ P and $-x \in$ P => $x = 0$.

Given a cone P⊆E, we define a Partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in \boldsymbol{P}$. We shall write x ≪ y to denote $x \leq y$ but $x \neq y$ to denote $y - x \in p^0$, where p^0 stands for the interior of P.
Remark 2.2 [7]: $\lambda p^0 \subseteq p^0$ for $\lambda > 0$ and $p^0 + p^0 \subseteq p^0$

Definition 2.2 [3] : Let X be a non-empty set and $d : X \times X \rightarrow E$ a mapping such that
(d_1) $0 \leq (x, y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,
(d_2) $d(x,y) = d(y,x)$ for all $x, y \in X$,
(d_3) $d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example 2.4 [3]: Let $E = R^2$, P $\{(x, y) \in E: x, y \geq 0\}$ and $X = Y$, defined by $d(x, y) = (\alpha| x - y |, \beta| x - y |, \gamma| x - y)$ where $\alpha, \beta, \gamma \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.5 [3]: Let (X, d) be a cone metric space, $x \epsilon X$ and $\{x_n\}$ be a sequence in X. Then
$(i) \{x_n\}_{n\geq 1}$ converges to x whenever to every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$.
$(ii) \{x_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$ there is a natural number N such that $d(x_n, x_m) \geq c$ for all $n, m \geq N$.
(iii) (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent in X.

Definition 2.6[3]: Let (X, d) $be\ a\ cone\ metric\ space, P\ be\ a\ cone\ in\ real\ Banach\ space\ E, if$
(i) $a \in$ P and $a \ll c$ for some $k \epsilon [0,1]$ then $a = 0$.
(ii) $a \in$ P and $a \ll c$ for some $k \epsilon [0,1]$ then $a = 0$.
(iii) $u \leq v, v \ll w$, then $u \ll w$.

Lemma 2.7
Let (x, d)*be a cone metric space and P* ***be*** *a cone metric space in real Banach space E and* $\alpha_1, \alpha_2, \alpha_3.\alpha_4\alpha \geq 0$. If $x_n \to x$, $y_n \to y$, $z_n \to z$, and
$a_n \to p$ *in x and* $\alpha_1\, d(x_{n,}\, x), +\alpha_2\, d(y_{n,}y) + \alpha_3 d(z_{n,}z) +.\,\alpha_4 d(p_{n,}p)$. *Then* $a = 0$.

III. Main Reault:

Theorem 3.1 Let (X, d)be a complete cone metric space with respect to a cone P containing in a real Banach space$E$. Let R_1, R_2 be any two surjective self mappings of X satisfy
$\boldsymbol{d}(R_1 x, R_2 y) \geq \alpha\, d(x, R_1 x) + \beta\, d(y, R_2 y) + \gamma\, d(x, y) + k[d(x, R_2 y) + d(y, R_1 x)]$…… (3.1.1)
for each $x,y \in X$, $x \neq y$ where α ,β, γ, $k \geq 0$, α + β + γ >1+2k, β + γ > k and γ >2k. Then R_1 and R_2 have a unique common fixed point.

Proof: Let x_0 be an arbitrary point in X. Since R_1 and R_2 surjective mappings, there exist points $x_1 \in R_1^{-1}(x_0)$ and $x_2 \in R_2^{-1}(x_1)$ that is $R_1(x_1) = x_0$ and $R_2(x_2) = x_1$. In this way, we define the sequence $\{x_n\}$ with $x_{2n+1} \in R_1^{-1}(x_{2n})$ and $x_{2n+2} \in R_2^{-1}(x_{2n+1})$.
i.e. $x_{2n} = R_1\, x_{2n+1}$ for n= 0,1 ,2,…………..(3.1.2)
$x_{2n+1} = R_2 x_{2n+2}$ for n =0,1,2……………(3.1.2)
Note that, if $x_{2n} = x_{2n+1}$ for some n≥0, then x_{2n} is fixed point of R_1 and R_2. Now putting x =x_{2n+1} and $y = x_{2n+2}$ from (3.1.1), we have
$d(R_1 x_{2n+1,} R_2 x_{2n+2}) = \alpha\, d(x_{2n+1,}\, R_1\, x_{2n+1}) + \beta\, d(\, x_{2n+2,} R_2\, x_{2n+2}) + \gamma\, d(x_{2n+1,} x_{2n+2})$
$+ k[d(x_{2n+1,}\, R_2\, x_{2n+2}) + d\, (x_{2n+2,}\, R_2 x_{2n+1})]$
$\Rightarrow\ d(x_{2n}, x_{2n+1}) \geq \alpha\, d(x_{2n+1,}\, x_{2n}) + \beta\, d(\, x_{2n+2,} x_{2n+1}) + \gamma\, d(x_{2n+1,} x_{2n+2})$
$+ k[d(x_{2n+1,}\ x_{2n+1}) + d(x_{2n+2,}\, x_{2n})]$
$\Rightarrow\ d(x_{2n}, x_{2n+1}) \geq \alpha d(x_{2n+1,}\, x_{2n}) + \beta\, d(\, x_{2n+2,} x_{2n+1}) + \gamma d(x_{2n+1,} x_{2n+2})$
$+ k[d(x_{2n+2,}\ x_{2n+1}) + d(x_{2n+1,}\, x_{2n})]$
$\Rightarrow d(x_{2n}, x_{2n+1}) \geq [\alpha + k] d(x_{2n+1,}\, x_{2n}) + [\beta + \gamma + k] d(x_{2n+1,} x_{2n+2})$
$\Rightarrow d(x_{2n+1,} x_{2n+2}) \leq \frac{1-(\alpha+k)}{\beta+\gamma+k}(x_{2n}, x_{2n+1})$………………..(3.1.4)
Where h =$\left[\frac{1-(\alpha+k)}{\beta+\gamma+k}\right] < 1$, [as α + β + γ >1+2k]
In general
$d(x_{2n}, x_{2n+1}) \leq h\, d(x_{2n-1}, x_{2n})$
$\Rightarrow d(x_{2n}, x_{2n+1}) \leq h^{2n}(x_{2n-1}, x_{2n})$…………………..(3.1.5)
So for every positive integer p, we have

$$\begin{aligned} d(x_{2n}, x_{2n+p}) &\leq d(x_{2n,} x_{2n+1}) + d\,(x_{2n+1}\, x_{2n+2}) + \ldots\ldots\ldots + d(x_{2n+p-1,} x_{2n+p}) \\ &\leq (\, h^{2n} + h^{2n+1} + \ldots\ldots\ldots\ldots + h^{2n+p-1})\, d\,(x_0, x_1) \\ . \quad &= h^{2n}(1 + h + h^2 + \cdots \ldots\ldots\ldots. + h^{2n+p-1})\, d(x_0, x_1) \\ &< \frac{h^{2n}}{1-h}\, \mathrm{d}(x_0, x_1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(3.1.6) \end{aligned}$$

Therefore $\{x_{2n}\}$ is a Cauchy sequence, which is complete space in X there exist $x^* \in X$ such that$x_{2n \to} x^*$. Since R_1 is surjective map, there exist a point y in X such that
$y \in R_1^{-1}(x^*)$. i.e. $x^* = R_1(y)$..........................(3.1.7)
Now consider $d(x_{2n,x^*}) = d\,(R_1 x_{2n+1,} y)$
$\geq\ \alpha d\,(x_{2n+1,} R_1 x_{2n+1}) + \beta d(y, R_1 y) + \gamma d(x_{2n+1}, y)\ +\ k\big[d(x_{2n+1,} R_1 y) +$
d y,R1x2n+1
$\Rightarrow d(x^*, x^*)\ \geq\ \alpha d\,(x^*, x^*) + \beta d(y, x^*) + \gamma d(x^*, y) +$
$k\big[d(x_{2n+1,} R_1 y) + d(y, R_1 x_{2n+1})\big]$
$\Rightarrow d(x^*, x^*)\ \geq\ \alpha d\,(x^*, x^*) + \beta d(y, x^*) + \gamma d(x^*, y) + k\big[d(x_{2n+1,} R_1 y) + d(y, R_1 x_{2n+1})\big]$
$\Rightarrow \quad 0 \geq (\beta + \gamma + k)\mathrm{d}(x^*, y)$
$\Rightarrow\ d(x^* y) \quad = 0$, as $(\beta + \gamma + k) > 0$
$\Rightarrow\ x^* = \mathrm{y}$……………………………………………………..(3.1.8)
Hence x^* is a fixed point of $R_{1.}$ as $R_1 y =\ x^* = y$. Now if z be another fixed point of R_1 ,
i.e. R_1z = z. Then
$(x^*, \mathrm{z}) = d\,(R_1 x^*, R_1 \mathrm{z})$
$\geq \alpha d(R_1 x^*, R_1 z) +\ \beta d(z, R_1 z) + \gamma d(x^*, z) + k[d(x^*, z) + \mathrm{d}\,(\mathrm{z}, x^*)]$
$= \mathrm{k}\,[\, d(x^*, z) + (\, z, x^*)] + \gamma d\,(x^*, z)(\mathrm{z}, R_1 z)$
$= (2k + \gamma)\ d(x^*, \mathrm{z})$

$\Rightarrow d(x^*,z) \le \frac{1}{c+2k} d(x^*,z)$
$\Rightarrow d(x^*,z) = 0$ as $c > 2k$ as and by proposition 2.6 (i).
$\Rightarrow x^* = z$. Therefore R_1 has a unique fixed point .Similarly it can be established that $R_2x^* = x^*$. Hence $R_1x^* = x^* = R_2x^*$.Thus x^* is the common fixed point of R_1 and R_2. These completed the proof of the theorem.

Corollary 3.2 Let (x, d) be a complete cone metric space with respect to a cone P containing in a real Banach space E. Let $R_1$and R_2 be any two surjective self mappings of X satisfying
$d(R_1x, R_2 y) \ge \alpha d(x, R_1x) + \beta d(y, R_2y) + \gamma d(x, y)$ ……..(3.1.9)
For each , $\in X$, $x \neq y$ where $\alpha, \beta, \gamma, \ge 0, \alpha + \beta + \gamma > 1$. Then R_1 and R_2 have a unique fixed point.

Proof: The proof of the corollary immediately follows by putting k = o in the previous theorem.

Corollary 3.3 Let (x, d) be a complete cone metric space with respect to a cone p containing in a real Banach space E. Let $R_1$and R_2 be any two surjective self mappings of X satisfying
$d(R_1x, R_2 y) \ge k[d(x, R_2y) + d(y, R_1x)]$ For each $x, \in X$, $x \neq y$ where $k \ge 0$
Then R_1 and R_2 have a unique fixed point.

Proof: The proof of the corollary immediately follows by putting $\alpha = 0, \beta = 0$ *and* $\gamma = 0$ in the previous theorem.

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