IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728. Volume 6, Issue 1 (Mar. - Apr. 2013), PP 19-23
www.iosrjournals.org

m - projective curvature tensor on a Lorentzian para – Sasakian manifolds

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Abstract: *In this paper we studied m-projectively flat, m-projectively conservative, φ-m-projectively flat LP-Sasakian manifold. It has also been proved that quasi m- projectively flat LP-Sasakian manifold is locally isometric to the unit sphere $S^n(1)$ if and only if M^n is m-projectively flat.*

Keywords – *Einstein manifold, m-projectively flat, m-projectively conservative, quasi m-projectively flat, φ-m-projectively flat.*

I. Introduction

The notion of Lorentzian para contact manifold was introduced by K. Matsumoto. The properties of Lorentzian para contact manifolds and their different classes, viz LP-Sasakian and LSP-Sasakian manifolds have been studied by several authors. In [13], M.Tarafdar and A. Bhattacharya proved that a LP-Sasakian manifold with conformally flat and quasi - conformally flat curvature tensor is locally isometric with a unit sphere $S^n(1)$. Further, they obtained that an LP-Sasakian manifold with $R(X,Y) \cdot \tilde{C} = 0$ is locally isometric with a unit sphere $S^n(1)$, where $\tilde{C}$ is the conformal curvature tensor of type (1, 3) and $R(X,Y)$ denotes the derivation of tensor of tensor algebra at each point of the tangent space. J.P. Singh [10] proved that an m-projectively flat para-Sasakian manifold is an Einstein manifold. He has also shown that if in an Einstein P-Sasakian manifold $R(\xi,X) \cdot W = 0$ holds, then it is locally isometric with a unit sphere $H^n(1)$. Also an n-dimensional η-Einstien P-Sasakian manifold satisfying $W(\xi,X) \cdot R = 0$ if and only if either manifold is locally isometric to the hyperbolic space $H^n(-1)$ or the scalar curvature tensor r of the manifold is $-n(n-1)$. S.K. Chaubey [18], studied the properties of m-projective curvature tensor in LP-Sasakian, Einstein LP-Sasakian and η-Einstien LP-Sasakian manifold. LP-Sasakian manifolds have also studied by Matsumoto and Mihai [4], Takahashi [11], De, Matsumoto and Shaikh [2], Prasad & De [9], Venkatesha and Bagewadi[14].

In this paper, we studied the properties of LP-Sasakian manifolds equipped with m-projective curvature tensor. Section 1 is introductory. Section 2 deals with brief account of Lorentzian para-Sasakian manifolds. In section 3, we proved that an m-projectively flat LP-Sasakian manifold is an Einstein manifold and an LP-Sasakian manifold satisfying $(C_1^1 W)(Y,Z) = 0$ is of constant curvature is m-projectively flat. In section 4, we proved that an Einstein LP-Sasakian manifold is m-projectively conservative if and only if the scalar curvature is constant. In section 5, we proved that an n-dimensional φ-m-projectively flat LP-Sasakian manifold is an η-Einstein manifold. In last, we proved that an n-dimensional quasi m - projectively flat LP-Sasakian manifold M^n is locally isometric to the unit sphere $S^n(1)$ if and only if M^n is m-projectively flat.

II. Preliminaries

An n- dimensional differentiable manifold M^n is a Lorentzian para-Sasakian (LP-Sasakian) manifold, if it admits a (1, 1) - tensor field ϕ, a vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$
$$\eta(\xi) = -1, \tag{2.2}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$
$$g(X,\xi) = \eta(X), \tag{2.4}$$
$$(D_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.5}$$
$$\text{and } D_X\xi = \phi X, \tag{2.6}$$

for arbitrary vector fields X and Y, where D denote the operator of covariant differentiation with respect to Lorentzian metric g, (Matsumoto, (1989) and Matsumoto and Mihai, (1988)).
In an LP-Sasakian manifold M^n with structure(ϕ,ξ,η,g), it is easily seen that

$(a)\ \phi\xi = 0 \quad (b)\ \eta(\phi X) = 0 \quad (c)\ rank\ \phi = (n-1)$ (2.7)

Let us put $F(X,Y) = g(\phi X, Y)$, (2.8)

then the tensor field F is symmetric (0, 2) tensor field

$$F(X,Y) = F(Y,X), \tag{2.9}$$

$$F(X,Y) = (D_X\eta)(Y), \tag{2.10}$$

and $(D_X\eta)(Y) - (D_Y\eta)(X) = 0$. (2.11)

An LP- Sasakian manifold M^n is said to be Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = kg(X,Y). \tag{2.12}$$

An LP- Sasakian manifold M^n is said to be $an\ \eta$ -Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{2.13}$$

for any vector fields X and Y, where α, β are the functions on M^n.

Let M^n be an n-dimensional LP-Sasakian manifold with structure (φ, ξ, η, g). Then we have (Matsumoto and Mihai, (1998) and Mihai, Shaikh and De,(1999)).

$$g(R(X,Y)Z, \xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \tag{2.14}$$

$$R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X, \qquad (2.15)(a)$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \qquad (2.15)(b)$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \qquad (2.15)(c)$$

$$S(X,\xi) = (n-1)\eta(X), \tag{2.16}$$

$$S(\varphi X, \varphi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{2.17}$$

for any vector fields X, Y, Z ; where $R(X,Y)Z$ is the Riemannian curvature tensor of type (1, 3). S is a Ricci tensor of type (0, 2), Q is Ricci tensor of type (1, 1) and r is the scalar curvature.

$g(QX,Y) = S(X,Y)$ for all X, Y.

m-projective curvature tensor W on an Riemannian manifold (M^n, g) $(n > 3)$ of type (1, 3) is defined as follows (G.P.Pokhariyal and R.S. Mishra (1971)).

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY], \tag{2.18}$$

so that $'W(X,Y,Z,U) \stackrel{\text{def}}{=} g(W(X,Y)Z,U) =' W(Z,U,X,Y)$.

On an n- dimensional LP-Sasakian manifold, the Concircular curvature tensor C is defined as

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{2.19}$$

Now, in view of $S(X,Y) = \frac{r}{n}g(X,Y)$, (2.18) becomes

$$W(X,Y)Z = C(X,Y)Z.$$

Thus, in an Einstein LP-Sasakian manifold, m-projective curvature tensor W and the concircular curvature tensor C concide.

III. m-projectively flat LP-Sasakian manifold

In this section we assume that $W(X,Y)Z = 0$.

Then from (2.18), we get

$$R(X,Y)Z = \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{3.1}$$

Contracting (3.1) with respect to X, we get

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{3.2}$$

Hence we can state the following theorem.

Theorem 3.1: Let M^n be an n-dimensional m-projectively flat LP-Sasakian manifold, then M^n be an Einstien manifold.

Contracting (2.18) with respect to X, we get

$$(C_1^1W)(Y,Z) = S(Y,Z) - \frac{1}{2(n-1)}[nS(Y,Z) - S(Y,Z) + rg(Y,Z) - g(Y,Z)], \tag{3.3}$$

$$Or,\ (C_1^1W)(Y,Z) = \frac{n}{2(n-1)}\left[S(Y,Z) - \frac{r}{n}g(Y,Z)\right], \tag{3.4}$$

where $(C_1^1W)(Y,Z)$ is the contraction of $W(X,Y)Z$ with respect to X.

If $(C_1^1W)(Y,Z) = 0$, then from (3.4), we get

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{3.5}$$

Using (3.5) in (3.1), we get

$$'R(X,Y,Z,W) = \frac{r}{n(n-1)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \tag{3.6}$$

Hence we can state the following theorem.

Theorem 3.2. An m-projectively flat LP-Sasakian manifold satisfying $(C_1^1W)(Y,Z) = 0$ is a manifold of constant curvature.

Using (3.5) and (3.6) in (2.18), we get

$$W(X,Y)Z = 0,$$

i.e. the manifold M^n is m-projectively flat.
Hence we can state the following theorem.
Theorem 3.3. An LP-Sasakian manifold (M^n, g) $(n > 3)$ satisfying $(C_1^1 W)(Y,Z) = 0$, is of constant curvature is m-projectively flat.

IV. Einstein LP-Sasakian manifold satisfying $(div\, W)(X,Y)Z = 0$

Definition 4.1. A manifold (M^n, g) $(n > 3)$ is called m-projectively conservative if (Hicks N.J.(1969)),

$$div(W) = 0, \tag{4.1}$$

where div denotes divergence.
Now differentiating (2.18) covariently, we get

$$(D_U W)(X,Y)Z = (D_U R)(X,Y)Z - \frac{1}{2(n-1)}[(D_U S)(Y,Z)X - (D_U S)(X,Z)Y + g(Y,Z)(D_W Q)X - g(X,Z)(D_W Q)Y]. \tag{4.2}$$

Which gives on contraction

$$div(W)(X,Y)Z = div(R)(X,Y)Z - \frac{1}{2(n-1)}[(D_X S)(Y,Z) - (D_Y S)(X,Z) + \quad g(Y,Z)div(Q)X - gX, ZdivQY. \tag{4.3}$$

But $div(Q) = \frac{1}{2}dr$, using in (4.3), we get

$$div(W)(X,Y)Z = div(R)(X,Y)Z - \frac{1}{2(n-1)}\Big[(D_X S)(Y,Z) - (D_Y S)(X,Z) + \frac{1}{2}\, g(Y,Z)dr(X) - 12gX, ZdrY. \tag{4.4}$$

But from (Eisenhart L.P.(1926)), we have

$$div(R)(X,Y)Z = (D_X S)(Y,Z) - (D_Y S)(X,Z). \tag{4.5}$$

Using (4.5) in (4.4), we get

$$div(W)(X,Y)Z = \frac{(2n-3)}{2(n-1)}[(D_X S)(Y,Z) - (D_Y S)(X,Z)] - \frac{1}{4(n-1)}[\, g(Y,Z)dr(X) - \, g(X,Z)dr(Y)]. \tag{4.6}$$

If LP-Sasakian manifold is an Einstein manifold, then from (1.12) and (4.5), we get

$$div(R)(X,Y)Z = 0. \tag{4.7}$$

From (4.6) and (4.7), we get

$$div(W)(X,Y)Z = -\frac{1}{4(n-1)}[\, g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{4.8}$$

From (4.1) and (4.8), we get

$$[\, g(Y,Z)dr(X) - g(X,Z)dr(Y)] = 0,$$

which shows that r is constant. Again if r is constant then from (4.8), we get

$$div(W)(X,Y)Z = 0.$$

Hence we can state the following theorem.
Theorem 4.1. An Einstein LP-Sasakian manifold (M^n, g) $(n > 3)$ is m-projectively conservative if and only if the scalar curvature is constant.

V. φ- m-projectively flat LP-Sasakian manifold

Definition 5.1. A differentiable manifold $(M^n, g), n > 3$, satisfying the condition

$$\varphi^2 W(\varphi X, \varphi Y)\varphi Z = 0, \tag{5.1}$$

is called φ-m - projectively flat LP-Sasakian manifold.(Cabrerizo, Fernandez, Fernandez and Zhen (1999)).

Suppose that $(M^n, g), n > 3$ is a φ - m - projectively flat LP-Sasakian manifold. It is easy to see that $\varphi^2 W(\varphi X, \varphi Y)\varphi Z = 0$, holds if and only if

$$g(W(\varphi X, \varphi Y)\varphi Z, \varphi W) = 0,$$

for any vector fields X, Y, Z, W.
By the use of (2.18), φ – m - projectively flat means

$$'R(\varphi X, \varphi Y, \varphi Z, \varphi W) = \frac{1}{2(n-1)}\begin{bmatrix} S(\varphi Y, \varphi Z)g(\varphi X, \varphi W) - S(\varphi X, \varphi Z)g(\varphi Y, \varphi W) + \\ g(\varphi Y, \varphi Z)S(\varphi X, \varphi W) - g(\varphi X, \varphi Z)S(\varphi Y, \varphi W) \end{bmatrix}, \tag{5.2}$$

where $'R(X,Y,Z,W) = g(R(X,Y)Z, W)$.

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n by using the fact that $\{\varphi e_1, \varphi e_2, \ldots, \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (5.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} {}'R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(\varphi Y, \varphi Z) g(\varphi e_i, \varphi e_i) - S(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) + g(\varphi Y, \varphi Z) S(\varphi e_i, \varphi e_i) - g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i)]. \tag{5.3}$$

On an LP-Sasakian manifold, we have ($\ddot{O}zg\ddot{u}$r (2003))

$$\sum_{i=1}^{n-1} {}'R(\varphi e_i, \varphi Y, \varphi Z, \varphi e_i) = S(\varphi Y, \varphi Z) + g(\varphi Y, \varphi Z), \tag{5.4}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, \varphi e_i) = r + (n-1), \tag{5.5}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) S(\varphi Y, \varphi e_i) = S(\varphi Y, \varphi Z), \tag{5.6}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi e_i) = (n+1), \tag{5.7}$$

$$\sum_{i=1}^{n-1} g(\varphi e_i, \varphi Z) g(\varphi Y, \varphi e_i) = g(\varphi Y, \varphi Z), \tag{5.8}$$

so, by the virtue of (5.4)−(5.8), the equation(5.3) takes the form

$$S(\varphi Y, \varphi Z) = \left[\frac{r}{n-1} - 1\right] g(\varphi Y, \varphi Z). \tag{5.9}$$

By making the use of (2.3) and (2.17) in (5.9), we get

$$S(Y, Z) = \left[\frac{r}{n-1} - 1\right] g(Y, Z) + \left[\frac{r}{n-1} - n\right] \eta(Y) \eta(Z).$$

Hence we can state the following theorem.

Theorem 5.1. Let M^n be an n-dimenstional $n > 3, \varphi$ – m - projectively flat LP-Sasakian manifold, then M^n is an η −Einstein manifold with constants $\alpha = \left[\frac{r}{n-1} - 1\right]$ and $\beta = \left[\frac{r}{n-1} - n\right]$.

VI. quasi m-projectively flat LP-Sasakian manifold

Definition 6.1. An LP-Sasakian manifold M^n is said to be quasi m-projectively flat, if

$$g(W(X, Y)Z, \varphi U) = 0, \tag{6.1}$$

for any vector fields X, Y, Z, U.

From (2.18), we get

$$g(W(X, Y)Z, \varphi U) = g(R(X, Y)Z, \varphi U) - \frac{1}{2(n-1)} [S(Y, Z) g(X, \varphi U) - S(X, Z) g(Y, \varphi U) + g(Y, Z) S(X, \varphi U) - g(X, Z) S(Y, \varphi U)]. \tag{6.2}$$

Let $\{e_1, e_2, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M^n by using the fact that $\{\varphi e_1, \varphi e_2, \ldots, \varphi e_{n-1}, \xi\}$ is also a local orthonormal basis, if we put $X = \varphi e_i$, $U = e_i$ in (5.2) and sum up with respect to i, then we have

$$\sum_{i=1}^{n-1} g(W(\varphi e_i, Y)Z, \varphi e_i) = \sum_{i=1}^{n-1} g(R(\varphi e_i, Y)Z, \varphi e_i) - \frac{1}{2(n-1)} \sum_{i=1}^{n-1} [S(Y, Z) g(\varphi e_i, \varphi e_i) - S(\varphi e_i, Z) g(Y, \varphi e_i) + g(Y, Z) S(\varphi e_i, \varphi e_i) - g(\varphi e_i, Z) S(Y, \varphi e_i)]. \tag{6.3}$$

On an LP-Sasakian manifold by straight forward calculation, we get

$$\sum_{i=1}^{n-1} {}'R(e_i, Y, Z, e_i) = \sum_{i=1}^{n-1} {}'R(\varphi e_i, Y, Z, \varphi e_i) = S(Y, Z) + g(\varphi Y, \varphi Z), \tag{6.4}$$

$$\sum_{i=1}^{n-1} S(\varphi e_i, Z) g(Y, \varphi e_i) = S(Y, Z) - (n-1) \eta(Y) \eta(Z). \tag{6.5}$$

Using (5.4), (5.7), (6.4), (6.5) in (6.3), we get

$$\sum_{i=1}^{n-1} g(W(\varphi e_i, Y)Z, \varphi e_i) = S(Y, Z) + g(\varphi Y, \varphi Z)$$

$$-\frac{1}{2(n-1)}[(n-1)S(Y,Z)+(r+n-1)g(Y,Z)+2(n-1)\eta(Y)\eta(Z)]. \qquad (6.6)$$

Using (2.3) in (6.6), we get

$$\sum_{i=1}^{n-1} g(W(\varphi e_i, Y)Z, \varphi e_i) = \frac{1}{2}\Big[S(Y,Z) - \Big(\frac{r}{n} - 1\Big) g(Y,Z)\Big]. \qquad (6.7)$$

If M^n is quasi m-projectively flat , then (6.7) reduces to

$$S(Y,Z) = \Big(\frac{r}{n} - 1\Big) g(Y,Z). \qquad (6.8)$$

Putting $Z = \xi$ in (6.8) and then using (2.6) and (2.16), we get

$$r = n(n-1). \qquad (6.9)$$

Using (6.9) in (6.8), we get

$$S(Y,Z) = (n-1)g(Y,Z). \qquad (6.10)$$

i.e. M^n is an Einstein manifold.

Now using (6.10) in (2.18), we get

$$W(X,Y)Z = R(X,Y)Z - [g(Y,Z)X - g(X,Z)Y]. \qquad (6.11)$$

If LP-Sasakian manifold is m-projectively flat , then from (6.11), we get

$$R(X,Y)Z = [g(Y,Z)X - g(X,Z)Y\ ]. \qquad (6.12)$$

Hence we can state the following theorem.

Theorem 6.2. A quasi m-projectively flat LP-Sasakian manifold M^n is locally isomeric to the unit sphere $S^n(1)$ if and only if M^n is m-projectively flat.

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