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Backlund's Theorem for Spacelike Surfacesin Minkowski 3-Space E_1^3

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Abstract: *By using the method of moving frames the Backlund's theorem and its applicationfor spacelike surfaces is introduced. The results leads to correspondence between the solutions of the Sine-Gordon equation and spacelike surfaces of constant positive Gaussian*
curvatures.
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I. Introduction

The construction of surfaces of negative constant Gaussian curvature in Euclidean space $E^3$is one of most important problems in differential geometry. The famous Backlund's theorem presented a geometrical method to construct a family of surfaces with Gaussian curvature $K=-1$ from a known surface with $K=-1$, .i. e., the Backlund transformation that we know well. On the other hand, it is well-known that there is a correspondence between solutions of the Sine-Gordon equation

$$\frac{\partial^2\varphi}{\partial u^2}-\frac{\partial^2\varphi}{\partial v^2}=\sin\varphi\ , \tag{1.1}$$

and the surfaces of constants negative Gaussian curvature in the Euclidean space $E^3$by realizing them as the focal surfaces of a pseudo-spherical $(p.s.)$ line congruence $[1,2,5]$.

Therefore the classical Backlund's theorem for such surfaces furnishes a way to generate new solutions of the Sine-Gordon equation from a given one.

In the theory of relativity, geometry of indefinite metric is very crucial. Hence, the theoryof surfaces in Minkowski space E_1^3which has the metric $ds^2=dx^2+dy^2-dz^2$ attracted much attention. A series of papers are devoted to the construction of surfaces of constant Gaussian curvature see for example $[1,2,6,7]$.

The situation is much more complicated than the Euclidean case, since the surfaces may have a definite metric (spacelike surfaces), Lorentz metric (timelike surfaces) or mixed metric.

The purpose of this paper is to study the class of spacelike surfaces with positiveGaussian curvature. It is seen that, as in the Euclidean case, the results leads to correspondence between the solutions of the Sine-Gordon equation and surfaces of constant positive Gaussian curvatures.

II. The Classical Theory

In the classical surfaces theory, a congruence of lines is a two-parameter set ofstraight lines. Such a congruence, has a parameterization in the form

$$\mathbf{y}(u,v,r)=\mathbf{p}(u,v)+r\mathbf{e}(u,v)\ ,\ \|\mathbf{e}\|=1\ , \tag{2.1}$$

where $\mathbf{p}(u,v)$ is its base surface (the surface of reference) and $\mathbf{e}(u,v)$ is the unit vector giving the direction of the straight lines of the congruence, r being a parameter on each line [3]. The equations

$$u=u(t),v=v(t),u'^2+v'^2\neq 0\ , \tag{2.2}$$

define a ruled surface belonging to the congruence. It is a developable if and only if the determinate

$$(\mathbf{e},d\mathbf{p},d\mathbf{e})=0\ . \tag{2.3}$$

This is a quadratic equation in du,dv. Suppose that it has two real and distinct roots, then there are two families of developables surfaces, each of which (in generic case) consists of the tangent lines of a surface. It

follows that the lines of the congruence are the common tangent lines of two surfaces M and M^*, to be called the focal surfaces. There results a mapping f:
$M \to M^*$ such that the congruence consists of the lines joining $\mathbf{p} \in M$ to $f(\mathbf{p}) \in M^*$. This simple construction plays a fundamental role in the theory of transformation of surfaces.
We rephrase this in more current terminology:

Definition (1.1): Consider a line congruence with focal surfaces M and M^* such that its lines are the common tangents at $\mathbf{p} \in M$ to $\mathbf{p}^* = f(\mathbf{p}) \in M^*$. The congruence is called a pseudo-spherical (p.s.) if
(i) $\|\mathbf{p}\mathbf{p}^*\| = r$, which is a constant independent of $\mathbf{p}$,
(ii) The angle between the normals at corresponding points is equal to a constant θ.
We can now state the classical Backlund's theorem:

Theorem (2.1)(*Backlund 1875*): Suppose there is a p.s. congruence in E^3 with the focalsurfaces M and M^* such that the distance r between corresponding points and the angle θ between corresponding normals are constants. Then both M and M^* have constant negative Gaussian curvature equal to $-sin^2\theta/r^2$.
There is also an integrability theorem:

Theorem (2.2): Suppose M is a surface in E^3 of constant negative Gaussian curvature $K = -sin^2\theta/r^2$, where $r > 0$ and $0 < \theta < \pi$ are constants. Given any unit vector $\mathbf{e} \in T(M_p)$, which is not a principal direction, there exists a unique surface M^* and a p.s. congruence $f: M \to M^*$ such that if $\mathbf{p}^* = f(\mathbf{p})$, we have $\mathbf{p}\mathbf{p}^* = r\mathbf{e}$ and θ is the angle between the normals at $\mathbf{p}$ and $\mathbf{p}^*$.
Thus one can construct one-parameter family of new surface of constant negative Gaussian's curvature from a given one, the results by varying r.

III. Backulnd's Theorem for Spacelike Surfaces

Let M be a spacelike surface in E_1^3. We choose a local field of orthonormal frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ $(<\mathbf{e}_1, \mathbf{e}_1>=<\mathbf{e}_2, \mathbf{e}_2>=-<\mathbf{e}_3, \mathbf{e}_3>=1)$ with origin $\mathbf{p}$ is a point of M and the vectors $\mathbf{e}_1, \mathbf{e}_2$ are tangent to M at $\mathbf{p}$.Let $\omega_1, \omega_2, \omega_3$ be the dual forms to the frame $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. We can write

$$d\mathbf{p} = \sum_\alpha \omega_\alpha \mathbf{e}_\alpha, d\mathbf{e}_\alpha \sum_\beta \omega_{\alpha\beta}\mathbf{e}_\beta . \tag{3.1}$$

Here and through this paper we shall agree on the index ranges

$$1 \le i, j, k \le 2, 1 \le \alpha, \beta, v \le 3 .$$

Note that:

$$\omega_{12} + \omega_{21} = 0, \omega_{31} = \omega_{13}, \omega_{32} = \omega_{23} . \tag{3.2}$$

The structure equations of E_1^3 are:

$$d\omega_\alpha = \sum_\beta \omega_\beta \wedge \omega_{\beta\alpha}, d\omega_{\alpha\beta} = \sum_v \omega_{\alpha v} \wedge \omega_{v\beta} . \tag{3.3}$$

Restricting these forms to the frame defined above. We have

$$\omega_3 = 0 , \tag{3.4}$$

and hence

$$0 = d\omega_3 = \sum_i \omega_i \wedge \omega_{i3} . \tag{3.5}$$

By Cartan's lemma we may write

$$\omega_{i3} = \sum_j h_{ij}\omega_j , h_{ij} = h_{ji} . \tag{3.6}$$

The first equation of (3.3) gives

$$d\omega_i = \sum_j \omega_j \wedge \omega_{ji} \,, \tag{3.7}$$

Where ω_{12} is the Levi-Civita connection form on M which is uniquely determined by these two equations.

The Gauss equation is

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{ki} + \Omega_{ij} \,, \tag{3.8}$$

where

$$\Omega_{12} = \omega_{13} \wedge \omega_{32} = \omega_{13} \wedge \omega_{23} = K\omega_1 \wedge \omega_2 \,, \tag{3.9}$$

$K = -det(h_{ij})$ is the Gaussian curvature of the surface M.

The Codazzi equations are:

$$d\omega_{i3} = \sum_j \omega_{ij} \wedge \omega_{j3} \,. \tag{3.10}$$

In the following theorems, we can now state the Minkowski versions of the Backlund's theorem:

Theorem (3.1): Suppose there is a $p.s.$spacelike line congruence $f$in E_1^3 between spacelike focal surfaces M,and M^*. If the distance $r$between corresponding points and the angle θ between corresponding normals$(< \mathbf{e}_3^*, \mathbf{e}_3 >= \cosh\theta)$ are constants. Then both M,and M^*have constant positive Gaussian curvature equal to $\sinh^2\theta/r^2$.

Proof. Since a $p.s.$spacelike line congruence exists between $M$and M^*then there is an orthonormal moving frame $\mathbf{e}_\alpha^*$ adapted to M^*on a neighborhood of$\mathbf{p}^* = f(\mathbf{p})$. Choose the direction vector along the congruence as

$$\mathbf{t} = \cos\gamma \mathbf{e}_1 + \sin\gamma \mathbf{e}_2 \,, \tag{3.11}$$

where $\gamma$is the angle between $\mathbf{e}_1$ and **t** on the tangent space of M.We can take the normal vector of M^*so that

$$< \mathbf{e}_3^*,\mathbf{e}_3 >= \operatorname{csch}\theta \,, < \mathbf{e}_3^*, \mathbf{e}_3^* >= -1, < \mathbf{e}_3^*,\mathbf{t} >= 0 \,, \tag{3.12}$$

where $\theta > 0$ is constant. The normal vector of M^* can be expressed as

$$\mathbf{e}_3^* = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - \cosh\theta\, \mathbf{e}_3 \,. \tag{3.13}$$

Substituting (3.13) into (3.12), then

$$x_1 = -\sinh\theta \sin\gamma \,, x_2 = \sinh\theta \cos\gamma \,. \tag{3.14}$$

Suppose locally M is given by an immersion $\mathbf{p}: U \to E_1^3$, where U is an open subset of the u, v plan, then M^* is given by

$$\mathbf{p}^* = \mathbf{p} + r\mathbf{t} \,. \tag{3.15}$$

Taking differentiation of (3.15) and using the structure equations, we get

$$d\mathbf{p}^* = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + r(d\gamma + \omega_{21})\mathbf{t}^\perp + r(\omega_{13}\cos\gamma + \omega_{23}\sin\gamma)\mathbf{e}_3 \,, \tag{3.16}$$

$$\mathbf{t}^\perp = -\sin\gamma \mathbf{e}_1 + \cos\gamma \mathbf{e}_2 \,, \tag{3.17}$$

And from $< d\mathbf{p}^*, \mathbf{e}_3^* >= 0$, we have

$$r(d\gamma + \omega_{21}) = \omega_1 \sin\gamma - \omega_2\cos\gamma - r\coth\theta\,(\omega_{13}\cos\gamma + \omega_{23}\sin\gamma) \,. \tag{3.18}$$

Again, taking differentiating of (3.18), and using the structure equations, we get

$$r\omega_{23\wedge}\omega_{31} = (d\gamma + \omega_{12}) \wedge [\omega_1 \cos\gamma + \omega_2 \sin\gamma - r\coth\theta\,(\omega_{23}\cos\gamma - \omega_{13}\sin\gamma)]\,. \tag{3.19}$$

Thus, by using (3.18) we obtain

$$r^2\omega_{23\wedge}\omega_{31} = [\omega_1 \sin\gamma - \omega_2\cos\gamma - r\coth\theta\,(\omega_{13}\cos\gamma + \omega_{23}\sin\gamma)] \wedge \left[\left(\omega_1\cos\gamma + \omega_2\sin\gamma - r\coth\theta\,(\omega_{23}\cos\gamma - \omega_{13}\sin\gamma)\right)\right], \tag{3.20}$$

or by means of (3.6) – (3.10), then (3.20) reduces to

$$K = \frac{\sinh^2\theta}{r^2}. \tag{3.21}$$

as claimed. By interchanging the roles of M, M^* in the above argument, we would obtain $K^* = \sinh^2\theta\,/r^2$ as well.

IV. Backlund's Transformations on Spacelike Surfaces

Now, we consider that $M$is spacelike surface with no umbilical points, i e. we can take the lines of curvature as its parametric curves. The first and second fundamental forms of $M$are:

$$\left.\begin{aligned} I &= A^2du^2 + B^2dv^2 \\ II &= k_1A^2du^2 + k_2B^2dv^2, \end{aligned}\right\} \tag{4.1}$$

where $h_{12} = h_{21} = 0$ and $h_{ii} = k_i$ are the principal curvatures of M. According to equation (3.6),we get that

$$\left.\begin{aligned} \omega_{13} &= k_1Adu, \omega_{23} = k_2Bdv, \\ \omega_1 &= Adu, \omega_2 = Bdv, \end{aligned}\right\} \tag{4.2}$$

Then the Codazzi equations can be written by

$$\left.\begin{aligned} (k_1 - k_2)A_v + (k_1)_vA = 0, \\ (k_2 - k_1)B_u + (k_2)_uB = 0. \end{aligned}\right\} \tag{4.3}$$

The Levi-Civita connection form will be

$$-\omega_{21} = \omega_{12} = -\frac{1}{B}A_vdu + \frac{1}{A}B_udv. \tag{4.4}$$

Now suppose Mhas Gaussian curvature $K \equiv 1$; then $k_1k_2 = -1$.By (4.3),we get

$$\left(\frac{k_1^2+1}{k_1}\right)A_v + (k_1)_vA = 0, \left(\frac{k_2^2+1}{k_2}\right)B_u + (k_2)_uB = 0. \tag{4.5}$$

Or

$$\frac{\partial}{\partial v}lin\left|A\sqrt{k_1^2+1} = 0, \frac{\partial}{\partial u}lin\right|B\sqrt{k_2^2+1} = 0. \tag{4.6}$$

Equations (4.6) means that, we can choose two positive valued functions $a(u), b(v)$ such that

$$A = \frac{a(u)}{\sqrt{k_1^2+1}}, B = \frac{b(v)}{\sqrt{k_2^2+1}}. \tag{4.7}$$

By making change in the coordinate system, we may assume that $a = b = 1$. Let $k_1 = \tan\frac{\varphi}{2}$,and since$k_1k_2 = -1$,so $k_2 = -\cot\frac{\varphi}{2}$, and $A = \cos\frac{\varphi}{2}$, $B = \sin\frac{\varphi}{2}$, so that fundamental forms are:

$$I = \cos^2\frac{\varphi}{2}du^2 + \sin^2\frac{\varphi}{2}dv^2, II = \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}(du^2 - dv^2). \tag{4.8}$$

It follows from (4.2) that

$$\omega_1 = \cos\frac{\varphi}{2}du, \omega_2 = \sin\frac{\varphi}{2}dv, \omega_{13} = \sin\frac{\varphi}{2}du, \omega_{23} = -\cos\frac{\varphi}{2}dv. \tag{4.9}$$

Thus we get

$$d\omega_{12} = \omega_{13}\wedge\omega_{32} = -\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}du\wedge dv, \tag{4.10}$$

in view of the Gauss's equation. And from equation (4.4), we have

$$d\omega_{12} = \frac{1}{2}\left(\frac{\partial^2\varphi}{\partial u^2} - \frac{\partial^2\varphi}{\partial v^2}\right) du \wedge dv\,. \tag{4.11}$$

From equations (4.10) and (4.11) it follows that

$$\frac{\partial^2\varphi}{\partial u^2} - \frac{\partial^2\varphi}{\partial v^2} = -2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\,. \tag{4.12}$$

Then

$$\frac{\partial^2\varphi}{\partial v^2} - \frac{\partial^2\varphi}{\partial u^2} = \sin\varphi\,. \tag{4.13}$$

We call (4.13) the Sine-Gordon equation in analogy with the classical Sine-Gordon equation (1.1). Since we have $K \equiv 1$, then $r = \sinh\theta$.Let $\gamma = -\varphi^*/2$, so equations (3.18), (4.4), and (4.9) gives

$$\left.\begin{aligned} \frac{1}{2}(\varphi_v - \varphi_u^*)\sinh\theta &= -\cos\frac{\varphi}{2}\sin\frac{\varphi^*}{2} - \cosh\theta\sin\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\,, \\ \frac{1}{2}(\varphi_u - \varphi_v^*)\sinh\theta &= -\sin\frac{\varphi}{2}\cos\frac{\varphi^*}{2} - \cosh\theta\cos\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\,. \end{aligned}\right\} \tag{4.14}$$

We call equations (4.14) the Backlund transformation.

Theorem (4.1): Suppose a $p.s.$spacelike congruence associated with spacelike focal surfaces $M$and M^*in E_1^3.Then both $M$and M^*have constant Gaussian's curvature$K \equiv +1$, and the angles $\varphi$and φ^*between their asymptotic directions are both solutions of the Sine-Gordon equation (4.13), and they are related by the Backlund transformation (4.14).

Proof. Recalling the definition of the vectorial product in E_1^3, we have

$$\begin{aligned} \mathbf{e}_3 \times \mathbf{e}_3^* &= \mathbf{e}_3 \times (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - \cosh\theta\,\mathbf{e}_3) \\ &= \sinh\theta\,(\cos\gamma\,\mathbf{e}_1 + \sin\gamma\,\mathbf{e}_2)\,. \end{aligned} \tag{4.15}$$

Then the vector **t** is tangent to both $M$and M^*. On the other hand, since (4.14)

$$\begin{aligned} \frac{1}{2}(\varphi_{vv} - \varphi_{uv}^*)\sinh\theta = \frac{1}{2}\varphi_v\left(\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2} - \cosh\theta\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\right) + \\ \frac{1}{2}\varphi_v^*\left(-\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2} + \cosh\theta\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\right), \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \frac{1}{2}(\varphi_{uu} - \varphi_{vu}^*)\sinh\theta = -\frac{1}{2}\varphi_u\left(-\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2} + \cosh\theta\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\right) + \\ \frac{1}{2}\varphi_u^*\left(\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2} + \cosh\theta\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\right). \end{aligned} \tag{4.17}$$

Then

$$\begin{aligned} \frac{1}{2}(\varphi_{vv} - \varphi_{uu})\sinh\theta = \frac{1}{2}(\varphi_v^* - \varphi_u)\left(-\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2} + \cosh\theta\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\right) + \\ \frac{1}{2}(\varphi_v - \varphi_u^*)\left(\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2} - \cosh\theta\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\right). \end{aligned} \tag{4.18}$$

Substituting (4.14) into (4.18), we have

$$\begin{aligned} 2(\varphi_{vv} - \varphi_{uu})\sinh^2\theta = \left(\sin\frac{\varphi}{2}\cos\frac{\varphi^*}{2} + \cosh\theta\cos\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\right)\times \\ \left(-\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2} + \cosh\theta\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2}\right) + \\ \left(-\cos\frac{\varphi}{2}\sin\frac{\varphi^*}{2} - \cosh\theta\sin\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\right)\times \\ \left(\sin\frac{\varphi}{2}\sin\frac{\varphi^*}{2} - \cosh\theta\cos\frac{\varphi}{2}\cos\frac{\varphi^*}{2}\right). \end{aligned} \tag{4.19}$$

It reduces to

$$\frac{\partial^2\varphi}{\partial v^2}-\frac{\partial^2\varphi}{\partial u^2}=\sin\varphi\ . \tag{4.20}$$

In a similar form, we can prove that φ^*satisfy the same Sine-Gordon equation, i.e.

$$\frac{\partial^2\varphi^*}{\partial v^2}-\frac{\partial^2\varphi^*}{\partial u^2}=\sin\varphi^*\ . \tag{4.21}$$

This complete the proof of the theorem.

For a $p.s.$spacelike line congruence, as we stated near a non-umbilical point on a spacelike surface M, we have a local frame field in which $\mathbf{e}1,\mathbf{e}2$ are along the principal directions. In addition, we have other local frame fields $\{\mathbf{p};\mathbf{t},\mathbf{t}^\perp,\mathbf{e}3\}$. Also we have a local frame field $\{\mathbf{p}^*;\mathbf{t}^*,\mathbf{t}^{*\perp},\mathbf{e}_3^*\}$ on M^*, where

$$\left.\begin{aligned}\mathbf{t}^*&:=-\mathbf{t}=-\cos\gamma\,\mathbf{e}_1-\sin\gamma\,\mathbf{e}_2\ ,\\ \mathbf{t}^{*\perp}&:=\mathbf{t}^*\times\mathbf{e}_3^*=\cosh\theta\,\mathbf{t}^\perp-\sinh\theta\,\mathbf{e}_3\ .\end{aligned}\right\} \tag{4.22}$$

Denote $\{\eta_1^*,\eta_2^*\}$ the dual forms to $\{\mathbf{p}^*;\mathbf{t}^*,\mathbf{t}^{*\perp},\mathbf{e}_3^*\}$ on M^*. By (3.16) and (3.18), we have

$$\begin{aligned}d\mathbf{p}^*=&\left(\cos\frac{\varphi^*}{2}\omega_1-\sin\frac{\varphi^*}{2}\omega_2\right)\mathbf{t}-r\coth\theta\left(\cos\frac{\varphi^*}{2}\omega_{31}-\sin\frac{\varphi^*}{2}\omega_{23}\right)\mathbf{t}^\perp\\ &+r\left(\cos\frac{\varphi^*}{2}\omega_{31}-\sin\frac{\varphi^*}{2}\omega_{23}\right)\mathbf{e}_3\ .\end{aligned} \tag{4.23}$$

Since $r=\sinh\theta$, we have

$$\eta_1^*:=-<d\mathbf{p}^*,\mathbf{t}^*>=\cos\frac{\varphi^*}{2}\cos\frac{\varphi}{2}du-\sin\frac{\varphi^*}{2}\sin\frac{\varphi}{2}dv\ , \tag{4.24}$$

and

$$\eta_2^*:=<d\mathbf{p}^*,\mathbf{t}^{\perp*}>=-\cos\frac{\varphi^*}{2}\sin\frac{\varphi}{2}du-\sin\frac{\varphi^*}{2}\cos\frac{\varphi}{2}dv\ . \tag{4.25}$$

To find out η_{31}^*, and η_{32}^* we take the differentiation of (3.13), and get

$$\begin{aligned}d\mathbf{e}_3^*=&-\sinh\theta\,(\omega_{!2}+d\gamma)\mathbf{t}-\cosh\theta\,(\omega_{31}\mathbf{e}_1+\omega_{32}\mathbf{e}_2)\\ &+\sinh\theta\left(\sin\frac{\varphi^*}{2}\omega_{31}+\cos\frac{\varphi^*}{2}\omega_{23}\right)\mathbf{e}_3\ .\end{aligned} \tag{4.26}$$

Therefore, we get

$$\eta_{31}^*:=-<d\mathbf{e}_3^*,\mathbf{t}^*>=\sin\frac{\varphi^*}{2}\cos\frac{\varphi}{2}du+\cos\frac{\varphi^*}{2}\sin\frac{\varphi}{2}dv\ , \tag{4.27}$$

$$\eta_{32}^*:=<d\mathbf{e}_3^*,\mathbf{t}^{*\perp}>=-\sin\frac{\varphi^*}{2}\sin\frac{\varphi}{2}du+\cos\frac{\varphi^*}{2}\cos\frac{\varphi}{2}dv\ . \tag{4.28}$$

Consider the local frames $\{\mathbf{p}^*;\ \mathbf{e}_1^*,\mathbf{e}_2^*,\mathbf{e}_3^*\}$,and $\{\mathbf{p}^*;\ \mathbf{t}^*,\mathbf{t}^{*\perp},\mathbf{e}_3^*\}$ on $M^*$are related by:

$$\left.\begin{aligned}\mathbf{t}^*&=\cos\frac{\varphi}{2}\mathbf{e}_1^*+\sin\frac{\varphi}{2}\mathbf{e}_2^*\ ,\\ \mathbf{t}^{*\perp}&=-\sin\frac{\varphi}{2}\mathbf{e}_1^*+\cos\frac{\varphi}{2}\mathbf{e}_2^*\ ,\end{aligned}\right\} \tag{4.29}$$

and denote its dual by $\{\omega_1^*,\omega_2^*\}$. Then, we get

$$\left.\begin{aligned}\omega_1^*&=\cos\frac{\varphi}{2}\eta_1^*-\sin\frac{\varphi}{2}\eta_2^*=\cos\frac{\varphi^*}{2}du\ ,\\ \omega_2^*&=\sin\frac{\varphi}{2}\eta_1^*+\cos\frac{\varphi}{2}\eta_2^*=-\sin\frac{\varphi^*}{2}dv\ ,\end{aligned}\right\} \tag{4.30}$$

and

$$\left.\begin{aligned}\omega_{31}^*&=\cos\frac{\varphi}{2}\eta_{31}^*-\sin\frac{\varphi}{2}\eta_{32}^*=\sin\frac{\varphi^*}{2}du\ ,\\ \omega_{32}^*&=\sin\frac{\varphi}{2}\eta_{31}^*+\cos\frac{\varphi}{2}\eta_{32}^*=\cos\frac{\varphi^*}{2}dv\ .\end{aligned}\right\} \tag{4.31}$$

Comparing the above formulas with (4.1), we see that $\mathbf{e}_1^*$, and $\mathbf{e}_2^*$ are the principal directions on the surface $M^*$and its fundamentals forms become:

$$\left.\begin{aligned} I^* &= \cos^2\frac{\varphi^*}{2}du^2 + \sin^2\frac{\varphi^*}{2}dv^2\,, \\ II^* &= \sin\frac{\varphi^*}{2}\cos\frac{\varphi^*}{2}(du^2 - dv^2)\,. \end{aligned}\right\} \qquad (4.32)$$

This means that $u, v$are Tschebyscheff coordinates on $M^*$and $\varphi^*$is its Tschebyscheff angle.
As a Minkowski version of integrability theorem, we may therefore state the following theorem:
Theorem (4.2): Suppose $M$is a spacelike surface with Gaussian curvature $K \equiv 1$ in E_1^3. For any given real number $\theta > 0$, we can construct a $p.s.$spacelike line congruence suchthat the solution of the completely integrable equation (4.14) is the Tschebyscheff angle ofthe corresponding surface M^*.
Proof. Let (u, v) be the Tschebyscheff coordinates on M. Then equation (4.14) is completely integrable by Theorem (4.1). Let φ^*be the solution of (4.14) such that $\varphi(u_0, v_0) = \varphi^*(u_0, v_0)$.Put $r = \sinh\theta$,then (4.9) hold. Let

$$\mathbf{t} = \cos\frac{\varphi^*}{2}\mathbf{e}_1 - \sin\frac{\varphi^*}{2}\mathbf{e}_2\,, \qquad (4.33)$$

and

$$\mathbf{p}^* = \mathbf{p} + r\mathbf{t}\,. \qquad (4.34)$$

We shall prove that $\mathbf{p}^*$ is a spacelike surface and the above congruence is a $p.s.$spacelike line congruence associated with $M$and M^*Suppose that

$$\mathbf{e}_3^* = \sinh\theta\left(\sin\frac{\varphi^*}{2}\mathbf{e}_1 + \cos\frac{\varphi^*}{2}\mathbf{e}_2\right) - \cosh\theta\,\mathbf{e}_3\,. \qquad (4.35)$$

By differentiation of (4.34), we get

$$d\mathbf{p}^* = -\sinh\theta\left[-\left(\frac{\varphi_v - \varphi_u^*}{2}\right)du + \left(\frac{\varphi_u - \varphi_v^*}{2}\right)dv\right]\left(\sin\frac{\varphi^*}{2}\mathbf{e}_1 + \cos\frac{\varphi^*}{2}\mathbf{e}_2\right)$$
$$+\sinh\theta\left(\cos\frac{\varphi^*}{2}\omega_{31} - \sin\frac{\varphi^*}{2}\omega_{23}\right)\mathbf{e}_3 + \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2. \qquad (4.36)$$

Then, we have

$$< d\mathbf{p}^*, \mathbf{e}_3^* >= 0\,, \qquad (4.37)$$

in view of (4.9) and (4.14). This mean that $\mathbf{e}_3^*$ is a normal vector of M^*,from (4.35) we have $< \mathbf{e}_3^*, \mathbf{e}_3^* >= -1$, i.e. M^* is a spacelike surface in E_1^3. Moreover, $< \mathbf{e}_3^*, \mathbf{e}_3 >= \cosh\theta = const.$, hence the line congruence is a p. s. spacelike line congruence. By Theorem (4.2), φ^* is the Tschebyscheff angle of M^*.This completes the proof of the theorem.
According to the above results, to construct a family of spacelike surfaces with $K = 1$from a known spacelike surface with $K = 1$, we only need to solve the completely integrableequations (4.14).

Bibliography

[1] **Abdel-Baky, R. A.;***The Backlund's Theorem in Minkowski 3-SpaceR_1^3, AMC 160, pp. 41-50, (2005).*

[2] **Chern, S. S. and Terng, C. L.;***An analogue of Backlund's Theorem in Affine Geometry, Rocky Mountain J. of Math.,V. 10. N.1, (1980).*

[3] **Chern, S. S.;***Geometrical interpretation of Sinh-Gordon equation, Ann. Polon. Math. 39, pp74-80, (1980).*

[4] **Chen, W.H.;***Some results on spacelike surfaces in Minkowski 3-Space, Acta. Math.Sincia, 37, 309-316, (1994)(In Chinese).*

[5] **Eisenhart, L. P.;***A Treatise in Differential Geometry of curves and surfaces, New York, Ginn Camp., (1969).*

[6] **Kobayashi, S. and Nomizu, K.;***Foundations of Differential Geometry, I, II New York, (1963), (1969).*

[7] **Mc-Nertney, L. V.;***One-Parameter families of surfaces with constant curvature in Lorentz three-space, Ph.D. Thesis, Brown University, (1980).*

[8] **Tian, C.;***Backlund transformations on surfaces with$K = -1$ in $R^{2,1}$, J. Geom. Phys. 22, pp.212-218, (1997).*

[9] **Zait, R. A.;***Backlund transformations and solutions for some Evolution equations, PhysicaScripta, Vol. 57, pp.545-548, (1998).*