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Compositions of the Generalized Operator $(G_{\rho,\eta,\gamma,\omega;a+}\Psi)(x)$ and their Application

Harish Nagar[1], Anil Kumar Menaria[2]
[1]*Department of Mathematics, Mewar University, Chittorgarh, India.*
[2]*Department of Mathematics, Mewar University, Chittorgarh, India.*

Abstract: *The present paper is devoted to the study of the compositions of operator of generalized function $\boldsymbol{G_{\rho,\eta,\gamma}[a,z]}$ defined in* [1] *and their applications. The compositions with Riemann-Liouville fractional integral and differential operator are also derived. As applications of our main-results some known results for generalized Mittag-Leffler function due to Kilbas et al.*[2] *are cited. The results involving the R-function* [3] *are also obtained as special cases of our main findings.*
Key words: *Generalized function, R-function, Generalized Mittag-Leffler function, Riemann-Liouville fractional calculus, Generalized fractional integral operators.*

I. Introduction and definitions:

The Riemann-Liouville fractional integral I_{a+}^{α} and fractional derivative D_{a+}^{α} of order α are defined by [4], [6]:

$$(I_{a+}^{\alpha}\,\Psi)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{\Psi(t)}{(x-t)^{1-\alpha}}\,dt, (\alpha\epsilon C, Re(\alpha)>0) \quad ..(1.1)$$

And

$$(D_{a+}^{\alpha}\,\Psi)(x)=\left(\frac{d}{dx}\right)^n (I_{a+}^{n-\alpha}\,\Psi)(x), \quad ..(1.2)$$

Where $(\alpha\epsilon C, Re(\alpha)>0; n=[Re(\alpha)]+1)$ respectively. The special G and R – Function are defined by [2,p.15,eqn.(10.1);4,p.1,eqn.(1.2)] (see also [3]:

$$G_{\rho,\eta,r}[a,z] = z^{r\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(r)_n(az^{\rho})^n}{\Gamma(n\rho+\rho r-\eta)\,n!}\ , Re(\rho r-\eta)>0, \quad ..(1.3)$$

At r=1, and z replaced by (z – c) it reduces to

$$R_{\rho,\eta}[a,c,z] = (z-c)^{\rho-\eta-1}\sum_{n=0}^{\infty}\frac{[a(z-c)^{\rho}]^n}{\Gamma(n\rho+\rho-\eta)}\ , \rho\geq 0, \rho\geq\eta, \quad ..(1.4)$$

The generalized Mittag-Leffler function defines by [4]:

$$E_{\rho,\mu}^{\gamma}(z) = \sum_{k=0}^{\infty}\frac{(\gamma)_k z^k}{\Gamma(\rho k+\mu)k!}\ , (\rho,\mu,\gamma\epsilon C, Re(\rho)>0, \quad ..(1.5)$$

Where at $\gamma=1$, $E_{\rho,\mu}^{1}(z)$ coincides with the classical Mittag-Leffler function $E_{\rho,\mu}(z)$ and in particular $E_{1,1}(z)=e^z$ and when $\rho=1$ it coincides with Kummer's confluent hyper geometric function $\emptyset(\gamma,\mu;z)$ with the exactness to the constant multiplier $[\ \Gamma(\mu)\ ]^{-1}$.

II. Some properties of function $G_{\rho,\eta,\gamma}[a,z]$

For $\rho,\eta,\gamma,\omega,\sigma,q,\alpha\epsilon C$, $(Re(\rho), Re(\eta), Re(q), Re(\alpha)>0)$ and $n\in N$ there hold the following properties for the special function $G_{\rho,\eta,\gamma}[a,z]$ defined in (1.3).

Property-1

$$\left(\frac{d}{dz}\right)^n[\,G_{\rho,\eta,\gamma}(\omega,z)] = G_{\rho,\eta+n,\gamma}(\omega,z), \quad ..(2.1)$$

Property-2

$$\int_0^x G_{\rho,\eta,\gamma}[\omega,(x-t)]\ G_{\rho,q,\sigma}[\omega,t]dt = G_{\rho,\eta+q,\gamma+\sigma}[\omega,x], \quad ..(2.2)$$

Property-3

$$I_{a+}^{\alpha}\big(\,G_{\rho,\eta,\gamma}[\omega,(t-a)]\big)(x) = G_{\rho,\eta-\alpha,\gamma}[\omega,(x-a)], (x>a), \quad ..(2.3)$$

Property-4

$$D_{a+}^{\alpha}\big(\,G_{\rho,\eta,\gamma}[\omega,(t-a)]\big)(x) = G_{\rho,\eta+\alpha,\gamma}[\omega,(x-a)], (x>a), \quad ..(2.4)$$

III. Compositions of the operator $(\mathbf{G}_{\rho,\eta,\gamma,\omega;a+}\boldsymbol{\Psi})(\mathbf{x})$ and the Inversion formula

Let $\rho,\eta,\gamma,\omega,\sigma,q,\alpha\epsilon C$, $(Re(\rho), Re(\eta), Re(q), Re(\alpha)>0)$ then the following results hold for $\Psi\in L(a,b)$.

Result-1

$$G_{\rho,\eta,\gamma,\omega;a+}\big[(t-a)^{\beta-1}\big](x) = \Gamma(\beta)\ G_{\rho,\eta-\beta,\gamma}[\omega,(x-a)], \quad ..(3.1)$$

Result-2

$$I_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+}\Psi = G_{\rho,\eta-\alpha,\gamma,\omega,a+}\Psi = G_{\rho,\eta,\gamma,\omega;a+} I_{a+}^{\alpha}\Psi\,, \quad ..(3.2)$$

Result-3

$$D_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+}\Psi = G_{\rho,\eta+\alpha,\gamma,\omega,a+}\Psi\,, \quad ..(3.3)$$

Holds for any continuous function $\Psi \in C[a\,,b]$.

Result-4

$$G_{\rho,\eta,\gamma,\omega;a+}\, G_{\rho,q,\sigma,\omega;a+}\Psi = G_{\rho,\eta+q,\gamma+\sigma,\omega,a+}\Psi\,, \quad ..(3.4)$$

Result-5

$$G_{\rho,\eta,\gamma,\omega;a+}\, G_{\rho,q,-\gamma,\omega;a+}\Psi = I_{a+}^{-(\eta+q)}\,\Psi, \quad ..(3.5)$$

Result-6

Let $G_{\rho,\eta,\gamma,\omega;a+}$ is invertible in the space L (a, b) and for $\Psi \in L(a\,,b)$,

$(G_{\rho,\eta,\gamma,\omega;a+}\Psi)(x) = f(x),\ a \le x \le b,$ then

$$\left\{\left[\,G_{\rho,\eta,\gamma,\omega;a+}\right]^{-1} f\right\}(x) = D_{a+}^{-(\eta+q)}\,(G_{\rho,\eta,-\gamma,\omega;a+}f)(x), \quad ..(3.6)$$

Outline of proof:

To prove the result in (4.1), we denote its LHS by Δ_4 i.e.

$$\Delta_4 = G_{\rho,\eta,\gamma,\omega;a+}\left[(t-a)^{\beta-1}\right](x)$$

Now making use of definition (1.2) and (1.3) and then term-by-term integration and with the help of Beta integral we at once arrive at the result (3.1) on using (1.3) therein.

To prove the result in (3.2), we denote its LHS by Δ_5 i.e. $\Delta_5 = \left(I_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+}\Psi\right)(x)$

Use the definition (1.1) and (1.3) and applying Dirichlet formula for $x > a$ we have:

$$\Delta_5 = \int_a^x \left(\left\{I_{a+}^{\alpha}\left[\,G_{\rho,\eta,\gamma}[\omega,\tau]\right]\right\}(x-t)\right)\Psi(t)dt$$

Now on using the relation (2.3) we at once arrive at the desired result in (3.2).The second relation of (3.2) is proved similarly.

To prove the result in (3.3), we denote its LHS by Δ_6 i.e. $\Delta_6 = (D_{a+}^{\alpha}\, G_{\rho,\eta,\gamma,\omega;a+}\Psi)(x)$

Now using the definition (1.2) and the result in (3.2) we have $\Delta_6 = \left(\frac{d}{dx}\right)^n (\,G_{\rho,\eta-n+\alpha,\gamma,\omega;a+}\Psi)(x)$

On applying (1.3) and (2.1) we at once arrive at the desired result in (3.3) in accordance with the definition (1.3).

To prove the result in (3.4), we denote its LHS by Δ_7 i.e. $\Delta_7 = \left(\,G_{\rho,\eta,\gamma,\omega;a+}\, G_{\rho,q,\sigma,\omega;a+}\Psi\right)(x)$

Now using the definition (1.3) we have

$$\Delta_7 = \int_a^x \left[\int_0^{x-t} G_{\rho,\eta,\gamma}[\omega, x-t-\tau]\, G_{\rho,q,\sigma}[\omega,\tau]\, d\tau\right]\Psi(t)dt$$

On evaluating the inner integral with the help of (2.2) and then with the help of (1.3) we at once arrive at the desired result in (3.4).

The result in (3.5) is obtained by taking $\sigma = -\gamma$ in (3.4) and in view of the relation (1.3).

To prove the inversion formula in (3.6), let $\left(\,G_{\rho,\eta,\gamma,\omega;a+}\Psi\right)(x) = f(x)$

Operating $G_{\rho,q,-\gamma,\omega;a+}$ on both the sides we have:

$$(G_{\rho,q,-\gamma,\omega;a+}\, G_{\rho,\eta,\gamma,\omega;a+}\Psi)(x) = (G_{\rho,q,-\gamma,\omega;a+}f)(x)$$

On using the result (3.5)

$$\left(I_{a+}^{-(\eta+q)}\,\Psi\right)(x) = \left(\,G_{\rho,q,-\gamma,\omega;a+}f\right)(x)$$

Now on operating $D_{a+}^{-(\eta+q)}$ on both the sides, it gives

$$\left(D_{a+}^{-(\eta+q)} I_{a+}^{-(\eta+q)}\,\Psi\right)(x) = \left(\,D_{a+}^{-(\eta+q)} G_{\rho,q,-\gamma,\omega;a+}f\right)(x) \text{ i.e.}$$

$$\Psi(x) = \left\{\left[\,G_{\rho,\eta,\gamma,\omega;a+}\right]^{-1} f\right\}(x) = \left(\,D_{a+}^{-(\eta+q)} G_{\rho,q,-\gamma,\omega;a+}f\right)(x)$$

Which is the result in (3.6).

Now let $\left\{\left[\,G_{\rho,\eta,\gamma,\omega;a+}\right]^{-1} f\right\}(x) = \left(\,D_{a+}^{-(\eta+q)} G_{\rho,q,-\gamma,\omega;a+}f\right)(x)$, then

$$\left\{\left[\,G_{\rho,\eta,\gamma,\omega;a+}\right]^{-1}\left[\,G_{\rho,\eta,\gamma,\omega;a+}\right] f\right\}(x) = D_{a+}^{-(\eta+q)} I_{a+}^{-(\eta+q)} f(x) = f(x)$$

This completes the proof of inversion formula (3.6).

IV. Applications

If we replace $\eta\ by\ \rho\gamma - \mu\ and\ q\ by\ \rho\sigma - \nu$ in the results (2.1) to (2.4) these reduce to the known result [2,pp. 36-39, eqs. (2.10),(2.21),(3.1),(3.2)] respectively which in turn at $\rho = 1$ provide the known [2, pp.36-39, eqs. (2.12), (2.25), (3.6), (3.7)] respectively.

If we replace η by $\rho\gamma - \mu$ and q by $\rho\sigma - \nu$ in the results (3.1) to (3.5) these reduce to the compositions for the fractional integral operator$\left(E^{\gamma}_{\rho,\mu,\omega;a+}\Psi\right)(x)$ for generalized Mittag-Leffler function in the kernel [2,pp.42-47,eqs.(4.15),(5.1),(5.5),(6.1),(6.5)]
Respectively which in turn at $\rho = 1$ provide the known [2, pp.43-47, eqs. (4.17), (5.3), (5.12), (6.4), (6.9)] respectively.
If in results (2.1) to (2.4) we take $\gamma = 1, \sigma = 1$,these results deduced to the following results involving $R_{\rho,\eta}[a, 0, z]$ function defined in (1.3).

$$\left(\frac{d}{dz}\right)^n \left[\, R_{\rho,\eta}[\omega, 0, z]\right] = \; R_{\rho,\eta+n}[\omega, 0, z], \quad ..\ (4.1)$$

$$\int_0^x \; R_{\rho,\eta}[\omega, 0, x-t] \; R_{\rho,q}[\omega, 0, t] = \; G_{\rho,\eta+q,2}[\omega, x], \quad ..(4.2)$$

$$I^{\alpha}_{a+}\left(\, R_{\rho,\eta}[\omega, a, t]\right)(x) = \; R_{\rho,\eta-\alpha}[\omega, a, x], (x > a), \quad ..\ (4.3)$$

$$D^{\alpha}_{a+}\left(\, R_{\rho,\eta}[\omega, a, t]\right)(x) = R_{\rho,\eta+\alpha}[\omega, a, x], (x > a), \quad ..\ (4.4)$$

If in results (3.1) to (3.4) we take $\gamma = 1, \sigma = 1$,these compositions deduce to the following compositions for the integral operator $R_{\rho,\eta,\gamma,\omega;a+}$ defined in (1.2)

$$\left[(t-a)^{\beta-1}\right](x) = \; \Gamma(\beta)\; R_{\rho,\eta-\beta}[\omega, 0, (x-a)], \quad ..(4.5)$$

$$R_{\rho,\eta,\omega;a+}\Psi = \; R_{\rho,\eta-\alpha,\omega,a+}\Psi = \; R_{\rho,\eta,\omega;a+} I^{\alpha}_{a+}\Psi, \quad ..(4.6)$$

$$D^{\alpha}_{a+}\; R_{\rho,\eta,\omega;a+}\Psi = \; R_{\rho,\eta+\alpha,\omega,a+}\Psi \quad ..(4.7)$$

$$R_{\rho,q,\omega;a+}\Psi = \; R_{\rho,\eta+q,2,\omega,a+}\Psi \quad ..(4.8)$$

If we replace η by $\rho - \mu$ and q by $\rho - \nu$ in the results 4.1) to (4.8) these reduce to the known results [2,pp.36-47,eqs.(2.11),(2.24),(3.4),(3.5),(4.16),(5.2),(5.10),(6.3)]
Respectively.

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