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Stability of Second Order Equation

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Abstract: *In this paper we discussed the stability of the null solution of the second order differential equation . Under some unusual assumptions we obtain new stability results for this classical equation.*

I. Introduction

Consider the second order ODE

$$x'' + 2f(t)x' + \beta(t)x + g(t)x^2 = 0,\ t \in R^+ \tag{1.1}$$

where $R^+ = [0, \infty)$, $f,\ \beta, g : R^+ \to R^+$ are three given continuous functions.

The most familiar interpretation of this equation is that it describes nonlinear oscillations. Stability problems for this ODE have been studied intensively so far (see, e.g.,[13]-[15], [16]-[18], and the references therein). Recently, T.A. Burton and T. Furumochi [5] have introduced a new method to study the stability of the null solution $x = x' = 0$ of equation (1.1), which is based on the Schauder fixed point theorem. They discussed a particular case of (1.1) (one of their assumptions is $\beta(t) = 1$) to illustrate their technique. In [8] Marosamu and Vladimirescu have proved stability results for the null solution of the same equation by using relatively classical arguments. Here, we reconsider Eq. (1.1) under more general assumptions, which require more sophisticated arguments, and prove stability results (see Theorem 2.1 below). In particular, we obtain the generalized exponential asymptotic stability of the trivial solution. See [20, p. 158] for the definition of this concept.

II. The main result

The following hypotheses will be required:

(i) $f \in C'(R^+)$ and $f(t) \geq 0$ for all $t \geq 0$

(ii) $\int_0^\infty f(t)dt = \infty$

(iii) there exist two constants $h, K \geq 0$ such that |

$$|f'(t) + f^2(t)| \leq Kf(t),\ t \in [h, \infty); \tag{2.1}$$

(iv) $\beta \in C'(\mathbb{R}^+)$, β decreasing, and

$$\beta(t) \geq \beta_0 > K^2 \text{, for all } t \in R^+ \tag{2.2}$$

where β_0 is constant.

(v) g is locally Lipschitzian in x

These assumptions are inspired by those in [5], but are more general.
Notice that (i) and (iii) imply that f is uniformly bounded (see [8], Remark 2.2).
The main result of this paper is the following theorem:

Theorem 2.1. If the assumptions (i) , (iii) - (v) are fulfilled, then the null solution of (1.1) is uniformly stable. If in addition (ii) holds, then the null solution of (1.1) is asymptotically stable

Remark 2.1. Under the assumptions (i) - (v), we cannot expect to have uniform asymptotic stability for the null solution. Indeed, even in the case $g = 0$ and $\beta =$ constant say $\beta(t) = 1\ \forall\ t \in R^+$ one can construct a fundamental matrix $X(t)$ for the corresponding first order linear differential system in $(x, y = x')$ for which

$$\|X(t)X(\tau)^{-1}\|$$

does not converge to zero as $t - \tau \to \infty$ (here $\|.\|$ denotes a matrix norm)

Proof of Theorem 2.1

we write equation (1.1) to a system

$$\boldsymbol{x' = y - f(t)x}$$
$$\boldsymbol{y' = \left(f'(t) + f^2(t) - 1\right)x - f(t)y - g(t)x^2}$$

and write it as

$$\boldsymbol{z' = \begin{pmatrix} -f(t) & 1 \\ -\beta(t) & -f(t) \end{pmatrix} X + \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ -g(t)x^3 \end{pmatrix}}$$

$$z' = A(t)z + B(t)z + F(t,z), \tag{2.3}$$

where

$$z = \begin{pmatrix} x \\ y \end{pmatrix},\ A(t) = \begin{pmatrix} -f(t) & 1 \\ -\beta(t) & -f(t) \end{pmatrix},\ \mathrm{B}(t) = \begin{pmatrix} 0 & 0 \\ f'(t) + f^2(t) & 0 \end{pmatrix}$$

$$F(t,z) = \begin{pmatrix} 0 \\ -g(t)x^2 \end{pmatrix}$$

It is easily seen that our stability question reduces to the stability of the null solution $z(t) = 0$ of system (2.3).

Let $t_0 \geq 0$ be arbitrarily fixed and let $z(t,t_0) = \begin{pmatrix} a(t,t_0) & b(t,t_0) \\ c(t,t_0) & d(t,t_0) \end{pmatrix},\ t \geq 0$ be the fundamental matrix to the linear system

$$z' = A(t)z \tag{2.4}$$

which is equal to the identity matrix for $t = t_0$ then

$$\begin{aligned} a'(t,t_0) &= -f(t)a(t,t_0) + c(t,t_0) \\ c'(t,t_0) &= -\beta(t)a(t,t_0) - f(t)c(t,t_0) \\ b'(t,t_0) &= -f(t)b(t,t_0) + d(t,t_0) \\ d'(t,t_0) &= -\beta(t)b(t,t_0) - f(t)d(t,t_0) \end{aligned} \tag{2.5}$$

So, since β is decreasing (hypothesis (iv)), the first two equations of (2.5) lead us to

$$\frac{1}{2}\left(\beta(t)a(t,t_0)^2 + c(t,t_0)^2\right)' \leq -f(t)\left(\beta(t)a(t,t_0)^2 + c(t,t_0)^2\right)$$

and hence

$$\beta(t)a(t,t_0)^2 + c(t,t_0)^2 \leq \beta(t_0)e^{-2\int_{t_0}^{t} f(u)du},\ \forall\, t \geq t_0. \tag{2.6}$$

Similarly, from the last two equations of (2.5), we get

$$\beta(t)b(t,t_0)^2 + d(t,t_0)^2 \leq e^{-2\int_{t_0}^{t} f(u)du},\ \forall\, t \geq t_0 \tag{2.7}$$

Consider $z = (x,y)^T \in R^2$ the norm $\|z\| = \sqrt{\beta_0 x^2 + y^2}$

For $z_0 = (x_0,y_0)^T \in R^2$ we obtain from (2.6),(2.7), and hypothesis (iv),

$$\begin{aligned} \|z(t,t_0)z_0\| &= \left\| \begin{matrix} a(t,t_0)x_0 + b(t,t_0)y_0 \\ c(t,t_0)x_0 + d(t,t_0)y_0 \end{matrix} \right\| \\ &\leq \sqrt{{x_0}^2 + {y_0}^2} \times \sqrt{\beta(t)[a(t,t_0)^2 + d(t,t_0)^2] + [c(t,t_0)^2 + d(t,t_0)^2} \\ &\leq \gamma\sqrt{1+\beta(}t_0)e^{-\int_{t_0}^{t} f(u)du}\|z_0\| \end{aligned} \tag{2.8}$$

where $\gamma = \max\{1, \frac{1}{\sqrt{\beta_0}}\}$ Moreover, since

$$z(t,t_0)z(s,t_0)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \eta(t,s,t_0) \\ \mu(t,s,t_0) \end{pmatrix},\ \forall\, t \geq s \geq t_0 \geq 0$$

Satisfies system (2.4), we deduce as before

$$\begin{aligned} \left\| z(t,t_0)z(s,t_0)^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| &= \sqrt{\beta_0\eta(t,s,t_0)^2 + \mu(t,s,t_0)^2} \\ &\leq \sqrt{\beta(t)\eta(t,s,t_0)^2 + \mu(t,s,t_0)^2} \\ &\leq \sqrt{\beta(s)\eta(t,s,t_0)^2 + \mu(t,s,t_0)^2}\, e^{-\int_s^t f(u)du} \\ &= e^{-\int_s^t f(u)du}, \quad t \geq s \geq t_0 \geq 0, \end{aligned} \tag{2.9}$$

Let us prove the first part of Theorem 2.1. Consider $z_0 \neq 0$ with $\|z_0\|$ small enough, $t_0 \geq 0$, and let us denote by $z(t,t_0 z_0)$ the unique solution of (2.3) which is equal to z_0 for $t = t_0$. By hypotheses (i) and (v), $z(t,t_0 z_0)$ is defined on a maximal right interval, say, $[t_0, l)$ and satisfies the following integral equation

$$z(t,t_0,z_0) = \mathrm{z}(\mathrm{t},t_0)z_0 + \int_{t_0}^{t} z(t,t_0)\, z(s,t_0)^{-1}\left[B(s)z(s,t_0,z_0) + F\left(s, z(t,t_0,z_0)\right)\right]ds \tag{2.10}$$

for all $t \in [t_0, l)$. From (2.8)-(2.10) we infer that

$$\begin{aligned} \|z(t,t_0,z_0)\| &\leq \gamma\sqrt{1+\beta(t_0)}\|z_0\|e^{-\int_{t_0}^{t} f(s)ds} \\ &\quad + \int_{t_0}^{t} e^{-\int_s^t f(u)du}\left[\left|f'(s) + f^2(s)\right||x(s,t_0,z_0)| + g(s)x^2(s)\right]ds, \quad \forall\, t \in [t_0, l) \end{aligned} \tag{2.11}$$

we infer from (2.11) that

$$\|\boldsymbol{z(t,t_0,z_0)}\| \leq \boldsymbol{\gamma\sqrt{1+\beta(t_0)}\|z_0\| + D\int_{t_0}^{t}\|z(s,t_0,z_0)\|ds,}\ \forall\, \boldsymbol{t} \in [\boldsymbol{t_0}, \boldsymbol{l}) \tag{2.12}$$

with a positive constant D and

$$\|\boldsymbol{z(t,t_0,z_0)}\| \leq \boldsymbol{\gamma\sqrt{1+\beta(t_0)}e^{Dh}\|z_0\|}\ , \forall\, \boldsymbol{t} \in [\boldsymbol{t_0}, \boldsymbol{l}) \tag{2.13}$$

Thus $z(t,t_0,z_0)$ as well as $z^{'}(t,t_0,z_0)$ are bounded on $[t_0,l)$ and so $z(t,t_0,z_0)$ can be extended to the right of l. This fact contradicts the maximality of l. Therefore $z(t,t_0,z_0)$ exists on $[t_0,l)$ with $l > h$.

Now, we assume $h < l < \infty$. We are going to find an estimate for $z(t,t_0,z_0)$ on the interval $[h,l)$. This time, our hypothesis (iii) comes into play. We have for all $t \in [h,l)$

$$\begin{aligned}\|\boldsymbol{z(t,t_0,z_0)}\| &\leq \boldsymbol{\gamma\sqrt{1+\beta(h)}\|z(h,t_0,z_0)\|e^{-\int_h^t f(u)du} + \int_h^t e^{-\int_s^t f(u)du} Kf(s)|x(s,t_0,z_0)|ds} \\ &+\boldsymbol{\int_h^t e^{-\int_s^t f(u)du} g(s)x^2(s,t_0,z_0)ds}\end{aligned} \tag{2.14}$$

From (2.14) it follows that

$$\begin{aligned}\|\boldsymbol{z(t,t_0,z_0)}\| &\leq \boldsymbol{\gamma\sqrt{1+\beta(h)}\|z(h,t_0,z_0)\|e^{-\int_h^t f(u)du} + \int_h^t e^{-\int_s^t f(u)du}\left|f^{'}(s)+f^2(s)\right||x(s,t_0,z_0)|ds} \\ &+\boldsymbol{\int_h^t e^{-\int_s^t f(u)du}\left|g(s)x^2(s,t_0,z_0)\right|ds} \\ &\leq \boldsymbol{\gamma\sqrt{1+\beta(h)}\|z(h,t_0,z_0)\|e^{-\int_h^t f(u)du}} \\ &+\boldsymbol{\int_h^t e^{-\int_s^t f(u)du}\left[Kf(s)|x(s,t_0,z_0)|ds + Mf(s)|x(s,t_0,z_0)|^3\right]ds} \\ &= \boldsymbol{\vartheta(t)}\ ,\ \forall\, \boldsymbol{t} \in [\boldsymbol{h}, \boldsymbol{l})\end{aligned} \tag{2.15}$$

Then by (2.15)

$$\boldsymbol{\vartheta^{'}(t)} \leq \left[\frac{\boldsymbol{K+M\vartheta(t)}}{\boldsymbol{\sqrt{\beta_0}}} - \boldsymbol{1}\right]\boldsymbol{\vartheta(t)f(t)} \quad \textbf{where } \boldsymbol{M} \in \left(\boldsymbol{0}, \boldsymbol{\sqrt{\beta_0} - K}\right)$$

$$\boldsymbol{\vartheta(h)} = \boldsymbol{\gamma\sqrt{1+\beta(h)}\|z(h,t_0,z_0)\|}$$

$$\|\boldsymbol{z(t,t_0,z_0)}\| \leq \boldsymbol{\vartheta(t)} \leq \boldsymbol{\vartheta(h)e^{\left(\frac{K+M\vartheta(h)}{\sqrt{\beta_0}}-1\right)\int_h^t f(u)du}},\ \forall \boldsymbol{t} \in [\boldsymbol{h}, \boldsymbol{l}] \tag{2.16}$$

From (2.16) we can see that $z(t,t_0,z_0)$ is bounded. Since $z^{'}(t,t_0,z_0)$ is also bounded, it follows that $l = \infty$. Now, for $\varepsilon > 0$ we denote

$$\boldsymbol{\delta = \delta(\varepsilon)} = \frac{\boldsymbol{\varepsilon e^{-Dh}}}{\boldsymbol{\gamma^2\sqrt{(1+\beta(0)(1+\beta(h)}}}$$

From (2.13) it follows that $\|z(t,t_0,z_0)\| \leq \frac{\varepsilon}{\gamma\sqrt{1+\beta(h)}}$ for all $t \in [t_0, h]$ provided that $\|z_0\| < \delta$. Therefore, by $M \in \left(0, \sqrt{\beta_0} - K\right)$ and (2.16), $\|z(t,t_0,z_0)\| \leq \vartheta(h) < \varepsilon$ for all $t \geq h$. Summarizing, if $0 \leq t_0 \leq h$, the solution $z(t,t_0,z_0)$ starting from any point, z_0, with $\|z_0\| < \delta$ exists on $[t_0,\infty)$ and satisfies $\|z(t,t_0,z_0)\| < \varepsilon$ for all $t \geq t_0$.

If $t_0 \geq h$, then analogously we obtain that $l = \infty$ and

$$\|\boldsymbol{z(t,t_0,z_0)}\| \leq \boldsymbol{\vartheta\sqrt{1+\beta(t_0)}\|z_0\|e^{\left(\frac{K+M\vartheta(h)}{\sqrt{\beta_0}}-1\right)\int_{t_0}^t f(u)du}},\ \forall \boldsymbol{t} \in [\boldsymbol{h}, \infty) \tag{2.17}$$

Therefore, with the same . as before, $\|z_0\| < \delta$ implies again that $\|z(t,t_0,z_0)\| < \varepsilon$ for all $t \geq t_0$ Hence the null solution is uniformly stable. If in addition (ii) is fulfilled, then by (2.16), and (2.17) it follows that the null solution to (1.1) is asymptotically stable. The proof of Theorem 2.1 is complete.

Remark 2.2 If f satisfies (i) - (iii), then $f(t) > 0$ for all $t \geq h$. Let us assume, by contradiction, that $f(t_1) = 0$ for some $t_1 \geq h$. Then, one can prove that $f(t) = 0$ for $t \geq t_1$ Indeed, if $f(t_2) > 0$ for some $t_2 > t_1$ then the function $u = \frac{1}{f}$ is well defined on the maximal interval containing t_2 on which $f > 0$, say (c,d), and satisfies the inequality

$$\boldsymbol{u^{'}(t) + Ku(t) - 1 \geq 0, t \in (c,d).}$$

This implies that

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left(\boldsymbol{e^{Kt}}\left[\boldsymbol{u(t)} - \frac{\boldsymbol{1}}{\boldsymbol{K}}\right]\right) \geq \boldsymbol{0, t \in (c, t_2]}$$

i.e., the function $t \to e^{Kt}\left[u(t) - \frac{1}{K}\right]$ decreasing on $(c, t_2]$, but this is impossible since the limit of this function is ∞ as $t \to c^{+}$. Thus, $f(t) = 0$ for all $t \geq t_1$, which contradicts (ii). Therefore, we have proved that indeed $f(t) > 0$ for all $t \geq h$ Consequently, the function $p(t) = \int_0^{\infty} f(s)ds$ is strictly increasing, at least for $t \geq h$. By the above proof we have that the null solution is generalized exponentially asymptotically stable with this $p(t)$.

Remark 2.3. if, $\beta(t) = 1$, $t \in R^+$ the fundamental matrix $Z(t, z_0)$ can be determined explicitly

$$\boldsymbol{z(t, z_0) = e^{-\int_{t_0}^{t} f(u)du} \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0 \\ -\sin(t - t_0 & \cos(t - t_0 \end{pmatrix}}$$

In general, this is not possible, so in our proof we had to get estimates without having an explicit form of $z(t, t_0)$

III. Conclusion :

These stability results can be extended to the case when the differential equation is vectorial equation. More precisely, assume that f and β are scalar functions satisfying the

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