Analysis on a Common Fixed Point Theorem

V. Srinivas¹, B.V.B. Reddy² & R. Umamaheshwar Rao ³

¹, ², ³ Department of Science and Humanities, Sreenidhi Institute of Science and Technology, Ghatkesar, Hyderabad, India - 501 301

Abstract: The aim of this paper is to prove a common fixed point theorem which generalizes the result of Brian Fisher [1] and et al. by weaker conditions. The conditions of continuity, compatibility and completeness of a metric space are replaced by weaker conditions such as reciprocally continuous and compatible, weakly compatible, and the associated sequence.

Keywords: Fixed point, self maps, reciprocally continuous, compatible maps, weakly compatible mappings.

I. Introduction

Two self maps S and T are said to be commutative if ST = TS. The concept of the commutativity has been generalized in several ways. For this Gerald Jungck [2] initiated the concept of compatibility.

1.1 Compatible Mappings.

Two self maps S and T of a metric space (X, d) are said to be compatible mappings if

\[ \lim_{n \to \infty} d(STx_n, TSx_n) = 0, \]

whenever \( <x_n> \) is a sequence in X such that \( \lim_{n \to \infty} Sx_n = Tx_n = t \) for some \( t \in X \).

It can be easily verified that when the two mappings are commuting then they are compatible but not conversely.

In 1998, Jungck and Rhoades [4] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely.

1.2. Weakly Compatible.

A pair of maps A and S is called weakly compatible pair if they commute at coincidence points.

Brian Fisher and others [1] proved the following common fixed Point theorem for four self maps of a complete metric space.

Theorem 1.3 Suppose A, B, S and T are four self maps of metric space(X, d) such that

1.3.1 (X, d) is a complete metric space
1.3.2 \( A(x) \subseteq T(x), B(x) \subseteq S(x) \)
1.3.3 The pairs (A, S) and (B, T) are compatible
1.3.4 \[ d(Ax, By)^2 < c_1 \max\{d(Sx, Ax)^2, d(Ty, By)^2, d(Sx, Ty)^2\} + c_2 \max\{d(Sx, Ax), d(Sx, By), d(Ax, Ty), d(By, Ty))\} + c_3 \max\{d(Sx, By), d(Ty, Ax)\} \]

where \( c_1, c_2, c_3 \geq 0, c_1 + 2c_2 < 1 \) and \( c_1 + c_3 > 1 \), then A, B, S and T have a unique common fixed point \( z \in X \).

1.4 Associated Sequence.

Suppose A, B, S and T are self maps of a metric space (X, d), satisfying the conditions (1.3.2). Then for any \( x_0 \in X, Ax_0 \in A(X) \) and hence, \( Ax_0 \in T(X) \) so that there is a \( x_1 \in X \) with \( Ax_1 = Tx_0 \). Now \( Bx_1 \in B(X) \) and hence there is \( x_2 \in X \) with \( Bx_2 = Sx_1 \). Repeating this process to each \( x_n \in X \), we get a sequence \( <x_n> \) in X such that

\[ Ax_{2n} = Tx_{2n+1} \quad \text{and} \quad Bx_{2n+1} = Sx_{2n+2} \quad \text{for} \quad n \geq 0. \]

We shall call this sequence as an associated sequence of \( x_0 \) relative to the Four self maps A, B, S and T.

Now we prove a lemma which plays an important role in proving our theorem.

1.5 Lemma. Suppose A, B, S and T are four self maps of a metric space (X, d) satisfying the conditions (1.3.2) and (1.3.4) of Theorem(1.3) and Further if (1.3.1) (X, d) is a complete metric space then for any \( x_0 \in X \) and for any of its associated sequence \( <x_n> \) relative to Four self maps, the sequence \( Ax_0, Bx_1, Ax_2, Bx_3, \ldots, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots, \) converges to some point \( z \in X \).
Analysis On A Common Fixed Point Theorem

Proof: For simplicity let us take \( d_n = d(y_n,y_{n+1}) \) for \( n=0,1,2, \ldots \).

We have

\[
d^2_{2n+1} = [d(y_{2n+1},y_{2n+2})]^2 = [d(Ax_{2n},Bx_{2n+1})]^2
\]

\[
\leq c_1 \max \{ d^2(Sx_{2n},Ax_{2n}), d^2(Tx_{2n+1},Bx_{2n+1}), d^2(Ax_{2n},Tx_{2n+1}), d^2(Bx_{2n},Tx_{2n+1}) \}
\]

\[
+ c_2 \max \{ d^2(Sx_{2n},Ax_{2n}), d^2(Sx_{2n},Bx_{2n+1}), d^2(Ax_{2n},Tx_{2n+1}), d^2(Bx_{2n},Tx_{2n+1}) \}
\]

\[
\leq c_1 \max \{ d^2(Ax_{2n},Bx_{2n+1}) \} + c_2 \{ d_{2n} \}
\]

\[
\leq c_1 \max \{ d^2(Ax_{2n},Bx_{2n+1}) \} + c_2 \{ d_{2n} \}
\]

\[
\leq c_1 \max \{ d^2(Ax_{2n},Bx_{2n+1}) \} + c_2 \{ d_{2n} \}
\]

If \( d_{2n+1} > d_{2n} \), inequality (1.5.1) implies \( d^2_{2n+1} \leq \frac{2c_2}{2c_2} \), a contradiction. Since \( \frac{3c_2}{2c_2} < 1 \), thus \( d_{2n+1} \leq d_{2n} \).

Similarly,

\[
d^2_{2n+1} = [d(y_{2n+1},y_{2n+2})]^2 = c_1 \max \{ d^2(Ax_{2n},Bx_{2n+1}) \} + c_2 \{ d_{2n} \}
\]

and it follows that

\[
d_{2n} = d(y_{2n},y_{2n+1}) \leq h d(y_{2n+1},y_{2n}) = h d_{2n}
\]

Consequently, \( d(y_{n+1},y_{n}) \leq h d(y_{n},y_{n-1}) \). For \( n=1,2,3, \ldots \), since \( h<1 \), this implies that \( \{y_n\} \) is a Cauchy sequence in \( X \).

Hence the Lemma.

The converse of the lemma is not true.

That is, suppose \( A, B, S \) and \( T \) are self maps of a metric space \( (X, d) \) satisfying the conditions (1.3.2) and (1.3.4), even for each associated sequence \( \langle x_n \rangle \) of \( x_0 \), the associated sequence converges, the metric space \( (X,d) \) need not be complete. For this we provide an example.

**1.6 Example.** Let \( X = (-1, 1) \) with \( d(x,y) = |x - y| \)

\[
Ax = Bx = \begin{cases}
\frac{1}{5} & \text{if } -1 < x < \frac{1}{6} \\
\frac{1}{6} & \text{if } \frac{1}{6} \leq x < 1 
\end{cases}
\]

\[
Sx = \begin{cases}
\frac{1}{5} & \text{if } -1 < x < \frac{1}{6} \\
\frac{6x + 5}{36} & \text{if } \frac{1}{6} \leq x < 1
\end{cases}
\]

\[
Tx = \begin{cases}
\frac{1}{5} & \text{if } -1 < x < \frac{1}{6} \\
\frac{1}{3} - x & \text{if } \frac{1}{6} \leq x < 1
\end{cases}
\]

Then \( A(X) = B(X) = \left[ \frac{1}{5}, \frac{1}{6} \right] \) while \( S(X) = \left[ \frac{1}{5}, \left[ \frac{1}{5}, \frac{1}{6}, \frac{11}{36} \right] \right] \) and \( T(X) = \left[ \frac{1}{5}, \left[ \frac{1}{5}, \frac{1}{6}, \frac{2}{3} \right] \right] \), so that \( A(X) \subset T(X) \) and \( B(X) \subset S(X) \) proving the condition (1.3.2) of Theorem (1.3). Clearly (X, d) is not a complete metric space.

It is easy to prove that the associated sequence \( A_{x_0}, B_{x_1}, A_{x_2}, B_{x_3}, \ldots, A_{x_{2n}}, B_{x_{2n+1}}, \ldots \) converges to \( \frac{1}{5} \) if \( -1 < x < \frac{1}{6} \) and converges to \( \frac{1}{6} \) if \( \frac{1}{6} \leq x < 1 \).

**II. Main Result**

**Theorem 2.** Suppose \( A, B, S \) and \( T \) are four self maps of metric space \( (X,d) \) such that

1. \( A(x) \subset T(x), B(x) \subset S(x) \),
2. The pair \( (A,S) \) is reciprocally continuous and compatible, and the pair \( (B,T) \) is weakly compatible.
3. \( d(Ax,Bx)^2 < c_1 \max \{ d(Sx,Ax)^2, d(Ty,Bx)^2, d(Sx,Ty)^2 \} + c_2 \max \{ d(Sx,Ax), d(Sx,Bx), d(Ty,Bx) \} \) where \( c_1, c_2, c_4 > 0 \) and \( c_2 + c_4 < 1 \) and \( c_2 + c_4 > 1 \).

www.iorsjournals.org 2 | Page
Further if

2.4 The sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_n, Bx_{2n+1}, \ldots$ converges to $z \in X$ then $A, B, S$ and $T$ have a unique common fixed point $z \in X$.

Proof: From condition IV, $Ax_{2n}$, $Bx_{2n+1}$ converges to $z$ as $n \to \infty$. Since the pair $(A, S)$ is reciprocally continuous means $ASx_{2n}$ converges to $Az$ and $SAx_{2n}$ converges to $Sz$ as $n \to \infty$.

Also since the pair $(A, S)$ is compatible, we get

$$\lim_{n \to \infty} d\left(ASx_{2n}, SAx_{2n}\right) = 0 \quad \text{or} \quad d(Az,Sz)=0 \quad \text{or} \quad Az= Sz .$$

Now $d(Az,z)^2 = d(Az,Bx_{2n+1})^2 \leq c_1 max \{d(Sz,Az)^2, d(Tx_{2n+1},Bx_{2n+1})^2, d(Sz,Bx_{2n+1})^2\} + c_2 \max \{d(Sz,Az),\ d(Az,Bx_{2n+1}),\ d(Az,Tx_{2n+1}),d(Bx_{2n+1},Tx_{2n+1})\}+c_3$

$$\{d(Sz,Bx_{2n+1}),d(Tx_{2n+1},Az)\}$$

Letting $n \to \infty$, we get

$$d(Az,z)^2 \leq c_1 d(Az,z)^2 + c_3 d(Az,z)^2 = (c_1+c_3) d(Az,z)^2$$

This gives $d(Az,z)^2[1-(c_1+c_3)] \leq 0$. Since $c_1+c_3 < 1$, we get $d(Az,z)^2=0$ or $Az=z$. Therefore $z=Az=Sz$.

Also since $A(x) \subseteq T(x) \ni u \in x$ such that $z=Az=Tu$.

We prove $Bz=Tu$. Consider $d((z,Bu)) = \{d(Az,Bu)^2 \leq c_1 \max \{d(Sz,Az)^2, d(Tu,Bu)^2, d(Sz,Tu)^2\} \varepsilon c_1 \max \{d(Sz,Bu), d(Az,Tu),d(Bu,Tz)\} + c_1 \{d(Sz,Bu),d(Tu,Az)\} = c_1 d(z,Bu)^2 + c_3 d(z,Bu)^2$

$$d(z,Bu)^2 \leq c_1 + c_3 d(z,Bu)^2$$

$d(z,Bu)^2[1-(c_1+c_3)] \leq 0$ since $c_1+c_3 < 1$, we get $d(z,Bu)^2=0$ or $Bu=Az$.

Therefore $z=Bu=Tu$.

Since the pair $(B, T)$ is weakly compatible and $z=Bu=Tu$, we get $d(BBu, TTu)=0$ or $Bz=Tu$.

Now consider $d(z,Bz)^2 = d(Az,Bz)^2 \leq c_1 \max \{d(Sz,Az)^2, d(Tz,Bz)^2, d(Sz,Tz)^2\} + c_3 \max \{d(Sz,Bz), d(Tz,Az),d(Bz,Tz)\} + c_1 \{d(Sz,Bz),d(Tz,Az)\} = c_1 d(z,Bz)^2 + c_3 d(z,Bz)^2$.

This gives

$$d(z,Bz)^2 \leq c_1 + c_3 d(z,Bz)^2$$

$d(z,Bz)^2[1-(c_1+c_3)] \leq 0$, since $c_1+c_3 < 1$, we get $d(z,Bz)^2=0$ or $z=Tu$.

Therefore $z=Bz=Tu$.

Since $z=Az=Bz=Sz=Tu$, $z$ is a common fixed point of $A, B, S$ and $T$.

The uniqueness of common fixed point can be easily proved.

Now, we discuss our earlier example in the following two remarks to justify our result.

Remark 2.5: From the example given earlier, clearly the pair $(A, S)$ is reciprocally continuous, since if $x_n=\left(\frac{1}{6} + \frac{1}{6n^2}\right)$ for $n \geq 1$, then

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = \frac{1}{6} \quad \text{and} \quad \lim_{n \to \infty} ASx_n = \frac{1}{6} = A(t) \quad \text{also} \quad \lim_{n \to \infty} SAx_n = \frac{1}{6} = S(t).$$

But none of $A$ and $S$ are continuous. Also, since

$$\lim_{n \to \infty} d(ASx_n,SAx_n)=0,$$

the pair $(A, S)$ is compatible. Moreover the pair $(B, T)$ is weakly compatible as they commute at coincident points $1 \quad \frac{1}{6} \quad \text{and} \quad \frac{1}{5} \quad \frac{1}{6}$. The contractive condition holds for the values of $c_1, c_2, c_3 \geq 0, c_1+2c_2 < 1$ and $c_1+c_3 > 1$. Further $\frac{1}{6}$ is the unique common fixed point of $A, B, S$ and $T$.

Remark 2.6: Finally we conclude that from the earlier example, the mappings $A, B, S$ and $T$ are not continuous, the pair $(A, S)$ is reciprocally continuous and compatible and $(B, T)$ is weakly compatible. Also the associated sequence relative to the self maps $A, B, S$ and $T$ such that the sequence $Ax_0, Bx_1, Ax_2, Bx_3, \ldots, Ax_{2n}, Bx_{2n+1}, \ldots$, converges to the point $\frac{1}{6} \in X$, but the metric space $X$ is not complete. Moreover, $\frac{1}{6}$ is the unique common fixed point of $A, B, S$ and $T$. Hence, Theorem (2) is a generalization of Theorem (1.3).
Analysis On A Common Fixed Point Theorem

References