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Expansion Maps In D-Metric And Tri D-Metric Spaces

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ABSTRACT *In this paper, we obtain some results on fixed points for expansion mappings in D-metric and Tri D-metric spaces, introduced by Dhage [1] .Our results includes several fixed point results in ordinary metric spaces as special cases on the line of Maia [5].*
KEYWORDS AND PHRASES: *Fixed point, D-metric spaces, Expansion maps, etc.*
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1. INTRODUCTION:

Motivated by the measure of nearness, the concept of a D-metric space introduced by Dhage [1] is as follows:

A nonempty set X together with a function $\rho : X \times X \times X \to [0,\infty)$, is called a D-metric space with a D-metric ρ , denoted by (X,ρ), if ρ satisfies the following properties:

(i) $\rho(x,y,z) = 0 \Leftrightarrow x = y = z$ (Coincidence) for all $x,y,z \in X$

(ii) $\rho(x,y,z) = \rho(\rho\{x,y,z\})$ (Symmetry) Where ρ is a permutation function.

(iii) $\rho(x,y,z) \le \rho(x,y,a) + \rho(x,a,z) + \rho(a,y,z)$ for all $x, y, z, a \in X$. (Tetrahedral inequality)

A sequence $\{x_n\} \subset X$, is said to be D-converges to a point $x \in X$ if $\lim_{m,n\to\infty} \rho(x_m, x_n, x) = 0$. Similarly , a sequence $\{x_n\} \subset X$, is called D-Cauchy if $\lim_{m,n,p\to\infty} \rho(x_m, x_n, x_p) = 0$. A complete D-metric space is one in which every D-Cauchy sequence converges to a point in it. A subset S of a D-metric space X is called bounded, if there exists a constant $K > 0$, such that $\rho(x,y,z) \le K$ for all $x, y, z \in S$. The infimum of all such k is called the diameter of S and is denoted by $\delta(S)$.

Let $f : X \to X$, then the orbit of f at a point $x \in X$ is a set in X , defined by $O_f(x) = \{x, fx, f^2x, \ldots.\}$. Again a D-metric space is called f-orbitally bounded if there exists a constant $M > 0$ such that $\rho(x,y,z) \le M$ for all $x, y, z \in O_f(x)$.A D-metric space is called f-orbitally complete if every D-Cauchy sequence in $O_f(x)$ converges to a point in X .

It is known that the D-metric ρ is a continuous function on X^3 in the topology of D-metric convergence which is Hausdorff, see Dhage [2].

In 1976, Rosenholtz [7] discussed local expansion mappings. Let (X,d) be an ordinary metric space. Then a mapping $T : X \to X$, expansive on a subset B of X , if $d(Tx,Ty) > d(x,y)$ for all $x, y \in B$ with $x \ne y$.

T is a Local expansion if every point in T has a neighbourhood B on which T is expansive.

In fact Rosenholtz proved, “If (X,d) be a complete metric space and $T : X \to X$ be a self map of X onto itself satisfying;

$d(Tx,Ty) > \lambda d(x,y)$ for all $x, y \in X$ with $x \ne y$ and $\lambda > 1$. Then T has a fixed point in X ”.

We need the following D-Cauchy principle developed by Dhage [3].

Lemma 1: (D-Cauchy principle): Let $\{x_n\}$ be bounded sequence with D-bound K, satisfying:
$(1.1.1)$ $\rho(y_n, y_{n+1}, y_p) \leq \lambda^n K$, for all $n, p \in N$ with $p > n$, where $0 \leq \lambda < 1$. Then $\{y_n\}$ is a D-Cauchy sequence.

Throughout in this paper we use the symbol
$$\rho(x, y, z).\rho(x, y, z) = \{\rho(x, y, z)\}^2 = \rho^2(x, y, z)$$

2. MAIN RESULTS:

THEOREM 2.1 : Let $f : X \to X$ be a surjective mapping of a f-orbitally bounded and f-orbitally complete D-metric space (X, ρ).If there exists non-negative reals $a_1, a_2, ... a_7$ with $a_1 + a_3 + a_5 > 0$, $a_2 < 1$ and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 > 1$, such that ;
$(2.1.1)$
$$\rho^2(fx, fy, fz) \geq a_1.\rho^2(x, y, z) + a_2.\rho^2(x, fx, fz) + a_3.\rho^2(y, fy, z) + a_4.\rho(x, fx, fz)\rho(x, y, z)$$
$$+ a_5.\rho(y, fy, z)\rho(x, y, z) + a_6.\rho(x, fx, fz)\rho(y, fy, z) + a_7.\rho(fx, fy, fz)\rho(x, y, z)$$
for all $x, y, z \in X$ with $x \neq y \neq z$.Then f has a fixed point in X.

PROOF: Let $x_0 \in X$. Since f is surjective, there exists an element x_1 satisfying $x_1 \in f^{-1}(x_0)$.By the way we can take $x_n \in f^{-1}(x_{n-1})$, $n = 2,3,4,.........$.
If $x_m = x_{m-1}$ for some m ,then x_m is a fixed point of f.
Without loss of generality, we can assume $x_n \neq x_{n-1}$ for every n .From (2.1.1), we have
$$\rho^2(x_{n-1}, x_n, x_{n+p-1}) = \rho^2(fx_n, fx_{n+1}, fx_{n+p})$$
$$\geq a_1.\rho^2(x_n, x_{n+1}, x_{n+p}) + a_2.\rho^2(x_n, fx_n, fx_{n+p}) + a_3.\rho^2(x_{n+1}, fx_{n+1}, x_{n+p})$$
$$+ a_4.\rho(x_n, fx_n, fx_{n+p}).\rho(x_n, x_{n+1}, x_{n+p}) + a_5.\rho(x_{n+1}, fx_{n+1}, x_{n+p}).\rho(x_n, x_{n+1}, x_{n+p})$$
$$+ a_6.\rho(x_6, fx_n, fx_{n+p}).\rho(x_{n+1}, fx_{n+1}, x_{n+p}) + a_7.\rho(fx_n, fx_{n+1}, fx_{n+p}).\rho(x_n, x_{n+1}, x_{n+p})$$
$$= a_1.\rho^2(x_n, x_{n+1}, x_{n+p}) + a_2.\rho^2(x_n, x_{n-1}, x_{n+p-1}) + a_3.\rho^2(x_{n+1}, x_n, x_{n+p})$$
$$+ a_4.\rho(x_n, x_{n-1}, x_{n+p-1}).\rho(x_n, x_{n+1}, x_{n+p}) + a_5.\rho(x_{n+1}, x_n, x_{n+p}).\rho(x_n, x_{n+1}, x_{n+p})$$
$$+ a_6.\rho(x_n, x_{n-1}, x_{n+p-1}).\rho(x_{n+1}, x_n, x_{n+p}) + a_7.\rho(x_{n-1}, x_n, x_{n+p-1}).\rho(x_n, x_{n+1}, x_{n+p})$$

Thus,
$$(a_1 + a_3 + a_5).\rho^2(x_n, x_{n+1}, x_{n+p}) + (a_4 + a_6 + a_7).\rho(x_{n-1}, x_n, x_{n+p-1}).\rho(x_n, x_{n+1}, x_{n+p})$$
$$- (1 - a_2).\rho^2(x_{n-1}, x_n, x_{n+p}) \leq 0$$
Or,
(2.1.2) $(a_1 + a_3 + a_5)t^2 + (a_4 + a_6 + a_7)t - (1 - a_2) \leq 0$, where
(2.1.3) $t = [\rho(x_n, x_{n+1}, x_{n+p}) / \rho(x_{n-1}, x_n, x_{n+p-1})]$
Let $g : [0, \infty) \to R$ be the function
(2.1.4) $g(t) = (a_1 + a_3 + a_5)t^2 + (a_4 + a_6 + a_7)t - (1 - a_2)$
Then from the hypothesis, $g(0) = a_2 - 1 < 0$
and $g(1) = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 - 1 > 0$.

Let $k \in (0,1)$ be the root of the equation $g(t) = 0$.
Then, $g(t) \le 0$ for $t \le k$ and therefore

$$\begin{aligned}\rho(x_n, x_{n+1}, x_{n+p}) &\le k.\rho(x_{n-1}, x_n, x_{n+p-1}) \\ &\le \mathrm{k}^2.\rho(x_{x-2}, x_{n-1}, x_{n+p-2}) \\ &\le \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\le k^n M\end{aligned}$$

Where M is a D-bound of $0_f(x)$.

Now, an application of Lemma 2.1 yields that $\{x_n\}$ is a D-Cauchy sequence. Since X is f-orbitally complete, there is a point $x \in X$ such that, $\lim_{n\to\infty} x_n = x$.
Now, we shall show that x is a fixed point of f.
Since, f is surjective there exists y in X, such that $y \in f^{-1}(x)$
For infinitely many $n, x_n \ne x$, hence for such n, we have

$$\begin{aligned}\rho^2(x_n, x, x) &= \rho^2(fx_{n+1}, fy, fy) \\ &\ge a_1.\rho^2(x_{n+1}, y, y) + a_2.\rho^2(x_{n+1}, fx_{n+1}, fy) + a_3.\rho^2(y, fy, y) \\ &+ a_4.\rho(x_{n+1}, fx_{n+1}, fy).\rho(x_{n+1}, y, y) + a_5.\rho(y, fy, y).\rho(x_{n+1}, y, y) \\ &+ a_6.\rho(x_{n+1}, fx_{n+1}, fy).\rho(y, fy, y) + a_7.\rho(fx_{n+1}, fy, fy).\rho(x_{n+1}, y, y) \\ &= a_1.\rho^2(x_{n+1}, y, y) + a_2.\rho^2(x_{n+1}, x_n, x) + a_3.\rho^2(y, x, y) \\ &+ a_4.\rho(x_{n+1}, x_n, x).\rho(x_{n+1}, y, y) + a_5.\rho(y, x, y).\rho(x_{n+1}, y, y) \\ &+ a_6.\rho(x_{n+1}, x_n, x).\rho(y, x, y) + a_7.\rho(x_n, x, x).\rho(x_{n+1}, y, y)\end{aligned}$$

On letting $n \to \infty$, we obtain

$$\begin{aligned}0 &\ge a_1.\rho^2(x, y, y) + a_2.\rho^2(x, x, x) + a_3.\rho^2(y, x, y) + a_4.\rho(x, x, x).\rho(x, y, y) \\ &+ a_5.\rho(y, x, y).\rho(x, y, y) + a_6.\rho(x, x, x).\rho(y, x, y) + a_7.\rho(x, x, x).\rho(x, y, y) \\ &= (a_1 + a_3 + a_5).\rho^2(x, y, y)\end{aligned}$$

Since, $a_1 + a_3 + a_5 > 0$, So $x = y$.
Thus x is a fixed point of f.
This completes the proof.

COROLLARY 2.2: Let $f : X \to X$ be a surjective mapping of a f-orbitally bounded and f-orbitally complete D-metric space X .If there exists a real constant $k > 1$, such that

(2.2.1) $\rho^2(fx, fy, fz) \ge k.\rho^2(x, y, z)$ for all $x, y, z \in X$ with $x \ne y \ne z$.

Then f has a fixed point in X.

PROOF : Proof of the corollary 2.2 follows easily from theorem 2.1.

3. It is possible that a D-metric space which is complete w.r.t. a D-metric but may not be complete w.r.t. another D-metric on X. In this section we consider a D-metric space with three D-metrics, i.e. a tri D-metric space and investigate some results on the fixed points on the line of Maia [5].

THEOREM 3.1: Let X be a D-metric space with three D-metrics ρ, ρ_1 and ρ_2. Let $f : X \to X$ be a surjective mapping. If there exists non-negative reals $a_1, a_2, a_3, \ldots\ldots a_7$ with $a_1 + a_3 + a_5 > 0$, $a_2 < 1$ and $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 > 1$, such that the following conditions hold in X ;

(i) $\rho_2(x,y,z) \le \rho_1(x,y,z) \le \rho(x,y,z)$ for all $x,y,z \in X$.

(ii) X is f-orbitally bounded and f-orbitally complete w.r.t. ρ_1

(iii) f is continuous w.r.t. ρ_2.

(iv) f Satisfies condition $(2.1.1)$ w.r.t. ρ.

Then f has a fixed point in X .

<u>PROOF</u>: Let $x_0 \in X$. Since f is surjective, there exists an element x_1 satisfying $x_1 \in f^{-1}(x_0)$.By the same way we can take

$$x_n \in f^{-1}(x_{n-1}) \ , \ n = 2,3,4,.........$$

Then proceeding as in the proof of theorem (2.1), with similar arguments , we get

$$\rho(x_n, x_{n+1}, x_{n+p}) \le k^n \rho(x_0, x_1, x_p)$$

Since, $\rho_1 \le \rho$ on X^3 , we have

$$\begin{aligned} \rho_1(x_n, x_{n+1}, x_{n+p}) &\le \rho(x_n, x_{n+1}, x_{n+p}) \\ &\le \mathrm{k}^n \rho(x_0, x_1, x_p) \\ &\le k^n M \text{ , where M is a D-bound of } 0_f(x) \text{ w.r.t. } \rho_1 \end{aligned}$$

Now, an application of Lemma 2.1 yields that $\{x_n\}$ is a D-Cauchy sequence in X . w.r.t. ρ_1. Since X is f-orbitally complete w.r.t. ρ_1 , there exists a point $x \in X$ such that,

$$\lim_{n\to\infty} x_n = x$$

Again since, $\rho_2 \le \rho_1$ on X^3 , we have

$$\lim_{n\to\infty} \rho_2{}^2(x_n, x, x) \le \lim_{n\to\infty} \rho_1{}^2(x_n, x, x) = 0$$

$$\text{Or}, \lim_{n\to\infty} \rho_2{}^2(x_n, x, x) = 0$$

This implies that the sequence $\{x_n\}$ converges to x w.r.t. ρ_2.

Now, by the continuity of f w.r.t. ρ_2 it follows that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} fx_{n+1} = f[\lim_{n\to\infty} x_n] = fx$$

Thus x is a fixed point of f . This completes the proof.

<u>COROLLARY</u> 3.2: Let X be a D-metric space with three D-metrics ρ, ρ_1 and ρ_2 .Let $f: X \to X$ be a surjective mapping. If there exists a real constant $k > 1$, such that, the following conditions hold in X ;

(i) $\rho_2(x,y,z) \le \rho_1(x,y,z) \le \rho(x,y,z)$ for all $x,y,z \in X$

(ii) X is f-orbitally bounded and f-orbitally complete w.r.t. ρ_1

(iii) f is continuous w.r.t. ρ_2.

(iv) f Satisfies condition $(2.2.1)$ w.r.t. ρ.

Then f has a fixed point in X .

<u>PROOF:</u> Proof of the corollary 3.2 follows easily from theorem 3.1.

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