Interval-Valued Fuzzy KUS-Ideals in KUS-Algebras

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Abstract: In this paper the notion of interval-valued fuzzy KUS-ideals (briefly i-v fuzzy KUS-ideal) in KUS-algebras is introduced. Several theorems are stated and proved. The image and inverse image of i-v fuzzy KUS-ideals are defined and how the homomorphic images and inverse images of i-v fuzzy KUS-ideals become i-v fuzzy KUS-ideals in KUS-algebras is studied as well.

Keywords: KUS-algebras, fuzzy KUS-ideals, interval-valued fuzzy KUS-sub-algebras, interval-valued fuzzy KUS-ideals in KUS-algebras.

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I. Introduction

W. A. Dudek and X. Zhang ([2],[3]) studied ideals and congruences of BCC-algebras. C. Prabpayak and U. Leerawat ([5],[6]) introduced a new algebraic structure which is called KU-algebra and investigated some related properties. The concept of a fuzzy set, was introduced by L.A. Zadeh [8]. O.G. Xi [7] applied the concept of fuzzy set to BCK-algebras and gave some of its properties. In [9], L.A. Zadeh made an extension of the concept of fuzzy set by an interval-valued fuzzy set (i.e., a fuzzy set with an interval-valued membership function). This interval-valued fuzzy set is referred to as an i-v fuzzy set. He constructed a method of approximate inference using his i-v fuzzy sets. In [1], R. Biswas defined interval-valued fuzzy subgroups and investigated some elementary properties. Recently S.M. Mostafa, and et al ([4]) introduced a new algebraic structure, called KUS-algebra, They have studied a few properties of these algebras, the notion of KUS-ideals on KUS-algebras was formulated and some of its properties are investigated. In this paper, using the notion of interval-valued fuzzy set by L.A. Zadeh, we introduce the concept of an interval-valued fuzzy KUS-ideals (briefly, i-v fuzzy KUS-ideals) of a KUS-algebra, and study some of their properties. Using an i-v level set of an i-v fuzzy set, we state a characterization of an i-v fuzzy KUS-ideals. We prove that every KUS-ideals of a KUS-algebra can be realized as an i-v level KUS-ideals of an i-v fuzzy KUS ideals of X. In connection with the notion of homomorphism, we study how the images and inverse images of i-v fuzzy KUS-ideals become i-v fuzzy KUS-ideals.

II. The Structure of KUS-algebras:

In this section we include some elementary aspects that are necessary for this paper.

Definition 2.1([4]). Let (X; *, .) be an algebra with a single binary operation (*). X is called a KUS-algebra if it satisfies the following identities:

(kus 1): (z * y) * (z * x) = y * x ,
(kus 2): 0 * x = x ,
(kus 3): x * x = 0 ,
(kus 4): x * (y * z) = y * (x * z) for any x, y, z ∈ X ,

In what follows, let (X; *, .) be a KUS-algebra. For brevity we also call X a KUS-algebra. In X we can define a binary relation (≤) by: x ≤ y if and only if y * x = 0 .

Lemma 2.2 ([4]). In any KUS-algebra (X; *, .), the following properties hold: for all x, y, z ∈ X;

a) x * y = 0 and y * x = 0 imply x = y ,
b) y * [(y * z) * z] = 0 ,
c) (0 * x) * (y * x) = y * 0 ,
d) x ≤ y implies that y * z ≤ x * z and z * x ≤ z * y ,
e) x ≤ y and y ≤ z imply x ≤ z ,
f) x * y ≤ z implies that y * z ≤ x .

Definition 2.3 ([4]). A nonempty subset I of a KUS-algebra X is called a KUS-ideal of X if it satisfies: for all x , y, z ∈ X , (Ikus 1) (0 ∈ I) .
(I(kus)) \((z \ast y) \in I \text{ and } (y \ast x) \in I \implies (z \ast x) \in I\).

**Definition 2.4**(8). Let \(X\) be a nonempty set, a fuzzy subset \(\mu \) in \(X\) is a function \(\mu : X \to [0,1]\).

**Definition 2.5**(4). Let \(X\) be a KUS-algebra and a fuzzy subset \(\mu \) in \(X\) is called a fuzzy KUS-sub-algebra of \(X\) if \(\mu(x \ast y) \geq \min\{\mu(x), \mu(y)\}\), for all \(x, y \in X\).

**Definition 2.6**(4). Let \(X\) be a KUS-algebra, a fuzzy subset \(\mu \) in \(X\) is called a fuzzy KUS-ideal of \(X\) if it satisfies the following conditions: for all \(x, y, z \in X\),

\[
\mu(z) = \min\{\mu(x), \mu(y)\},
\]

is called a level subset of \(\mu\).

**Theorem 2.9**(4). A fuzzy subset \(\mu\) of KUS-algebra \(X\) is a fuzzy KUS-ideal of \(X\) if and only if, for every \(t \in [0,1]\), \(\mu_t\) is either empty or a KUS-ideal of \(X\).

**Definition 2.10**(6). Let \((X; *, 0)\) and \((Y; *, 0')\) be nonempty sets. The mapping \(f : (X; *, 0) \to (Y; *, 0')\) is called a homomorphism if it satisfies

\[
f(x \ast y) = f(x) \ast' f(y) \text{ for all } x, y \in X.
\]

The set \(\{x \in X \mid f^{-1}(y) = 0\}'\) is called the Kernel of \(f\) and is denoted by \(\text{Ker } f\).

**Definition 2.11**(6). Let \(f : (X; *, 0) \to (Y; *, 0')\) be a mapping from the set \(X\) to a set \(Y\). If \(\mu\) is a fuzzy subset of \(X\), then the fuzzy subset \(\beta\) of \(Y\) defined by:

\[
f(\mu)(y) = \begin{cases} 
\sup\{\mu(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) = \{x \in X \mid f(x) = y\} \neq \emptyset \\
0 & \text{otherwise}
\end{cases}
\]

is said to be the image of \(\mu\) under \(f\).

Similarly if \(\beta\) is a fuzzy subset of \(Y\), then the fuzzy subset \(\mu = (\beta \circ f)\) in \(X\) (i.e. the fuzzy subset defined by \(\mu(x) = \beta(f(x))\) for all \(x \in X\)) is called the pre-image of \(\beta\) under \(f\).

**Theorem 2.12**(4). An into homomorphic pre-image of a fuzzy KUS-ideal is a fuzzy KUS-ideal.

**Theorem 2.13**(4). An into homomorphic image of a fuzzy KUS-ideal is a fuzzy KUS-ideal.

### III. Interval-Valued Fuzzy KUS-Ideals in KUS-Algebras

**Remark 3.1**(9). An interval-valued fuzzy subset (briefly i-v fuzzy subset) \(A\) defined in the set \(X\) is given by \(A = \{(x, [\mu^L_A(x), \mu^U_A(x)])\}\), for all \(x \in X\). (briefly, it is denoted by \(A = [\mu^L_A, \mu^U_A]\)) where \(\mu^L_A\) and \(\mu^U_A\) are any two fuzzy subsets in \(X\) such that \(\mu^L_A(x) \leq \mu^U_A(x)\) for all \(x \in X\).

Let \(\tilde{\mu}_A(x) = [\mu^L_A(x), \mu^U_A(x)]\), for all \(x \in X\) and let \(D[0,1]\) be denotes the family of all closed sub-interval of \([0,1]\). It is clear that if \(\mu^L_A(x) = \mu^U_A(x) = c\), where

\[0 \leq c \leq 1,\]

then \(\tilde{\mu}_A(x) = [c, c] \in D[0,1]\), then \(\tilde{\mu}_A(x) \in [0,1]\), for all \(x \in X\). Therefore the i-v fuzzy subset \(A\) is given by:

\[A = \{(x, \tilde{\mu}_A(x))\}, \text{ for all } x \in X \text{ where } \tilde{\mu}_A : X \to D[0,1].\]

Now we define the refined minimum (briefly \(r\) min) and order “\(\leq\)” on elements
\[D_1 = [a_1, b_1] \text{ and } D_2 = [a_2, b_2] \text{ of } D[0,1]\] as follows:
\[r \text{ min } (D_1, D_2) = \min\{a_1, a_2\}, \text{ min } \{b_1, b_2\}\] (1) \(D_1 \leq D_2 \iff a_1 \leq a_2\) and \(b_1 \leq b_2\). Similarly we can define \((\geq)\) and \((=)\).

In what follows, let \(X\) denote a KUS-algebra unless otherwise specified, we begin with the following definition.

**Definition 3.2.** An i-v fuzzy subset \(A\) in \(X\) is called an i-v fuzzy KUS-sub-algebra of \(X\) if \(\tilde{\mu}_A(x \ast y) \geq r \text{ min } (\tilde{\mu}_A(x), \tilde{\mu}_A(y))\), for all \(x, y \in X\).

**Example 3.3.** Let \(X = \{0, 1, 2, 3\}\) in which the operation \((\ast)\) be define by the following table:
Then (X; *, 0) is a KUS-algebra. Define a fuzzy subset μ: X → [0,1] by

\[ \mu(x) = \begin{cases} 
0.7 & \text{if } x = \{0,1\} \\
0.3 & \text{otherwise}
\end{cases} \]

I = [0,1] is a KUS-ideal of X. Routine calculation given that μ is a fuzzy KUS-ideal of X. Define \( \tilde{μ}_A \) (x) as follows:

\[ \tilde{μ}_A(x) = \begin{cases} 
[0.3,0.9] & \text{if } x = \{0,1\} \\
[0,1,0.6] & \text{otherwise}
\end{cases} \]

It is easy to check that A is an i-v fuzzy KUS-sub-algebra.

**Proposition 3.4.** If A is an i-v fuzzy KUS-sub-algebra of X, then \( \tilde{μ}_A(0) \geq \tilde{μ}_A(x) \), for all \( x \in X \).

**Proof.** For all \( x \in X \), we have \( \tilde{μ}_A(0) = \tilde{μ}_A(x \ast x) \geq r \min \{ \tilde{μ}_A(x), \tilde{μ}_A(x) \} \)

\[ = r \min \{ [μ_L A(x), μ_U A(x)] \} \leq r \min \{ [μ_L A(x), μ_U A(x)] \} = \tilde{μ}_A(x) \].

**Proposition 3.5.** Let A be an i-v fuzzy KUS-sub-algebra of X, if there exist a sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} \tilde{μ}_A(x_n) = [1,1] \), then \( \tilde{μ}_A(0) = [1,1] \).

**Proof.** By proposition (3.4), we have \( \tilde{μ}_A(0) \geq \tilde{μ}_A(x) \), for all \( x \in X \). Then \( \tilde{μ}_A(0) \geq \tilde{μ}_A(x_n) \), for every positive integer n. Consider the inequality

\[ [1,1] \geq \tilde{μ}_A(0) \geq \lim_{n \to \infty} \tilde{μ}_A(x_n) = [1,1] \]. Hence \( \tilde{μ}_A(0) = [1,1] \).

**Definition 3.6.** An i-v fuzzy subset A = \( \{x, \tilde{μ}_A(x)\} \), \( x \in X \) in KUS-algebra X is called an interval-valued fuzzy KUS-ideal (i-v fuzzy KUS-ideal, in short) if it satisfies the following conditions:

A1. \( \tilde{μ}_A(0) \geq \tilde{μ}_A(x) \).

A2. \( \tilde{μ}_A(z \ast x) \geq r \min \{ \tilde{μ}_A(z \ast y), \tilde{μ}_A(y \ast x) \} \), for all \( x, y, z \in X \).

**Example 3.7.** Let \( X = \{0,1,2,3\} \) as in example (3.3). Define \( \tilde{μ}_A(x) \) as follows:

\[ \tilde{μ}_A(x) = \begin{cases} 
[0.3,0.9] & \text{if } x = \{0,1\} \\
[0,1,0.6] & \text{otherwise}
\end{cases} \]

It is easy to check that A is an i-v fuzzy KUS-ideal of X.

**Theorem 3.8.** An i-v fuzzy subset A = \([μ_L A, μ_U A]\) in X is an i-v fuzzy KUS-ideal of X if and only if \( μ_L A \) and \( μ_U A \) are fuzzy KUS-ideals of X.

**Proof.** If \( μ_L A \) and \( μ_U A \) are fuzzy KUS-ideals of X. For any \( x, y, z \in X \). Observe \( \tilde{μ}_A(z \ast x) = [μ_L A(z \ast x), μ_U A(z \ast x)] \)

\[ \geq [\min \{ μ_L A(z \ast y), μ_L A(y \ast x)\}, \min \{ μ_U A(z \ast y), μ_U A(y \ast x)\}] \]

\[ = r \min \{ [μ_L A(z \ast y), μ_U A(y \ast x)] \} = \tilde{μ}_A(z \ast x) \] for all \( x, y, z \in X \).

From what was mentioned above we can conclude that A is an i-v fuzzy KUS-ideal of X.

Conversely, suppose that A is an i-v fuzzy KUS-ideal of X. For all \( x, y, z \in X \) we have \( μ_L A(z \ast x) \), \( μ_U A(z \ast x) \) = \( \tilde{μ}_A(z \ast x) \) \( \geq r \min \{ \tilde{μ}_A(z \ast y), \tilde{μ}_A(y \ast x) \} \)

\[ = r \min \{ [μ_L A(z \ast y), μ_U A(y \ast x)] \} = \tilde{μ}_A(z \ast x) \]
\[
\begin{align*}
= \left[ \min \{ \mu_A^L (z \ast y), \mu_A^L (y \ast x) \} \right], \min \{ \mu_A^U (z \ast y), \mu_A^U (y \ast x) \}. \therefore \mu_A^L (z \ast x) \geq \\
\min\{ \mu_A^L (z \ast y), \mu_A^L (y \ast x) \} \text{ and }
\mu_A^U (z \ast x) \geq \min\{ \mu_A^U (z \ast y), \mu_A^U (y \ast x) \}.
\end{align*}
\]

Hence, we get that \( \mu_A^L \) and \( \mu_A^U \) are fuzzy KUS-ideals of \( X \). \( \Box \)

**Theorem 3.9.** Let \( A_1 \) and \( A_2 \) be i-v fuzzy KUS-ideals of a KUS-algebra \( X \). Then \( A_1 \cap A_2 \) is an i-v fuzzy KUS-ideal of \( X \).

**Proof.** \( \tilde{\mu}_{A_1 \cap A_2} (0) = [ \mu_{A_1 \cap A_2}^L (0), \mu_{A_1 \cap A_2}^U (0) ] \geq [ \mu_{A_1 \cap A_2}^L (x), \mu_{A_1 \cap A_2}^U (x) ] = \tilde{\mu}_{A_1 \cap A_2} (x) \).

Suppose \( x, y, z \in X \) such that \( (z \ast y) \in A_1 \cap A_2 \) and \( (y \ast x) \in A_1 \cap A_2 \).

Since \( A_1 \) and \( A_2 \) are i-v fuzzy KUS-ideals of \( X \), then by the theorem (3.8), we get

\[
\tilde{\mu}_{A_1 \cap A_2} (z \ast x) = [ \mu_{A_1 \cap A_2}^L (z \ast x), \mu_{A_1 \cap A_2}^U (z \ast x) ]
= \left[ \min\{ \mu_{A_1 \cap A_2}^L (z \ast y), \mu_{A_1 \cap A_2}^L (y \ast x) \}, \min\{ \mu_{A_1 \cap A_2}^U (z \ast y), \mu_{A_1 \cap A_2}^U (y \ast x) \} \right]
= \left[ \min\{ \mu_{A_1 \cap A_2}^L (z \ast y), \mu_{A_1 \cap A_2}^U (y \ast x) \}, \min\{ \mu_{A_1 \cap A_2}^U (y \ast x), \mu_{A_1 \cap A_2}^U (z \ast x) \} \right]
= \mu_{A_1 \cap A_2} (z \ast x).
\]

**Corollary 3.10.** Let \( \{ A_i \}_{i \in I} \) be a family of i-v fuzzy KUS-ideals of \( X \). Then

\[ \bigcap_{i \in I} A_i \] is also an i-v fuzzy KUS-ideal of \( X \).

**Theorem 3.11.** Let \( X \) be a KUS-algebra and \( A \) be an i-v fuzzy subset in \( X \). Then \( A \) is an i-v fuzzy KUS-ideal of \( X \) if and only if the nonempty set

\[ \tilde{U} (A ; \delta_1, \delta_2) = \{ x \in X | \tilde{\mu}_A (x) \geq [\delta_1, \delta_2] \} \text{ is a KUS-ideal of } X \text{, for every} \]

\([\delta_1, \delta_2] \in D[0, 1] \text{. We call } \tilde{U} (A ; \delta_1, \delta_2) \text{ the i-v level KUS-ideal of } A.\]

**Proof.** Assume that \( A \) is an i-v fuzzy KUS-ideal of \( X \) and let \([\delta_1, \delta_2] \in D[0, 1] \) be such that \((z \ast y), (y \ast x) \in \tilde{U} (A ; [\delta_1, \delta_2]) \). Then

\[
\tilde{\mu}_A (z \ast x) \geq r \min\{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \} \geq r \min\{ [\delta_1, \delta_2], [\delta_1, \delta_2] \} = [\delta_1, \delta_2] \text{ and so } (z \ast x) \in \tilde{U} (A ; [\delta_1, \delta_2]).
\]

Conversely, assume that \( \tilde{U} (A ; [\delta_1, \delta_2]) \neq \emptyset \) is a KUS-ideal of \( X \), for every \([\delta_1, \delta_2] \in D[0, 1] \). In the contrary, suppose that there exist \( x_0, y_0, z_0 \in X \), such that

\[
\tilde{\mu}_A (z_0 \ast x_0) < r \min\{ \tilde{\mu}_A (z_0 \ast y_0), \tilde{\mu}_A (y_0 \ast x_0) \}.
\]

Let \( \tilde{\mu}_A (z_0 \ast y_0) = [\gamma_1, \gamma_2], \tilde{\mu}_A (y_0 \ast x_0) = [\gamma_3, \gamma_4] \) and \( \tilde{\mu}_A (z_0 \ast x_0) = [\delta_1, \delta_2]. \) If \( \tilde{\mu}_A (z_0 \ast y_0) = [\gamma_1, \gamma_2] < r \min\{ \gamma_1, \gamma_2 \}, \gamma_3, \gamma_4 \) = min\{ \gamma_1, \gamma_2 \}, \gamma_3, \gamma_4 \). So \( \delta_1 < \gamma_1, \gamma_2 \) and \( \delta_2 < \gamma_3, \gamma_4 \). Consider

\[
[\lambda_1, \lambda_2] = \frac{1}{2} \{ \tilde{\mu}_A (z_0 \ast x_0) + r \min\{ \tilde{\mu}_A (z_0 \ast y_0), \tilde{\mu}_A (y_0 \ast x_0) \} \}
\]

We find that

\[
[\lambda_1, \lambda_2] = \frac{1}{2} \{ (\delta_1, \delta_2) + r \min\{ [\gamma_1, \gamma_2], [\gamma_3, \gamma_4] \} \}
= \frac{1}{2} \{ (\delta_1 + \min\{ \gamma_1, \gamma_2 \}, (\delta_2 + \min\{ \gamma_3, \gamma_4 \}) \}.
\]

Therefore \( \min\{ \gamma_1, \gamma_2 \} > \lambda_1 = \frac{1}{2} (\delta_1 + \min\{ \gamma_1, \gamma_2 \}) > \delta_1 \).
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\[ \min \{\gamma_2, \gamma_4\} > \lambda_2 = \frac{1}{2} (\delta_2 + \min\{\gamma_2, \gamma_4\}) > \delta_2. \]

Hence \[ \min \{\gamma_1, \gamma_3\}, \min \{\gamma_2, \gamma_4\} > [\lambda_1, \lambda_2] > [\delta_1, \delta_2] = \tilde{\mu}_A (z_0 \ast x_0), \]

so that, \((z_0 \ast x_0) \notin \tilde{U} (A ; [\lambda_1, \lambda_2])\), which is a contradiction, since

\[ \tilde{\mu}_A (z_0 \ast y_0) = [\gamma_1, \gamma_3] \geq \min \{\gamma_1, \gamma_3\}, \min \{\gamma_2, \gamma_4\} > [\lambda_1, \lambda_2]. \]

\[ \tilde{\mu}_A (y_0 \ast x_0) = [\gamma_3, \gamma_4] \geq \min \{\gamma_1, \gamma_3\}, \min \{\gamma_2, \gamma_4\} > [\lambda_1, \lambda_2], \]

imply that \((z_0 \ast y_0) \ast (y_0 \ast x_0) \in \tilde{U} (A ; [\lambda_1, \lambda_2])\). Then

\[ \tilde{\mu}_A (z \ast y) \geq r \min \{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \}, \text{ for all } x, y, z \in X. \]

**Theorem 3.12.** Every KUS-ideal of a KUS-algebra X can be realized as an i-v level KUS-ideal of an i-v fuzzy KUS-ideal of X.

**Proof.** Let Y be a KUS-ideal of X and let A be an i-v fuzzy subset on X defined by

\[ \tilde{\mu}_A (x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in y \\ [0,0] & \text{otherwise} \end{cases}. \]

Where \(\alpha_1, \alpha_2 \in [0,1]\) with \(\alpha_1 < \alpha_2\). It is clear that \(\tilde{U} (A ; [\alpha_1, \alpha_2]) \in Y\). We show that A is an i-v fuzzy KUS-ideal of X. Let \(x, y, z \in X\).

If \((z \ast y), (y \ast x) \in Y\), then \((z \ast x) \in Y\), and therefore

\[ \tilde{\mu}_A (z \ast x) = [\alpha_1, \alpha_2] = r \min \{ [\alpha_1, \alpha_2], [\alpha_1, \alpha_2] \} = r \min \{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \}. \]

If \((z \ast y), (y \ast x) \notin Y\), then \(\tilde{\mu}_A (z \ast y) = [0,0] = \tilde{\mu}_A (y \ast x)\) and so

\[ \tilde{\mu}_A (z \ast x) \geq [0,0] = r \min \{ [0,0], [0,0] \} = r \min \{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \}. \]

Similarly for the case \((z \ast y) \notin Y\) and \((y \ast x) \in Y\) we get

\[ \tilde{\mu}_A (z \ast x) \geq r \min \{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \}. \]

Therefore A is an i-v fuzzy KUS-ideal of X. The proof is complete. □

**Proposition 3.13.** Let X be a KUS-algebra, B be a fuzzy subset on X and let A be an i-v fuzzy subset on X defined by \(\tilde{\mu}_A (x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in y \\ [0,0] & \text{otherwise} \end{cases}\). Where

\(\alpha_1, \alpha_2 \in (0,1]\) with \(\alpha_1 < \alpha_2\). If A is an i-v fuzzy KUS-sub-algebra of X, then B is a fuzzy KUS-sub-algebra of X.

**Proof.** Clear. □

**Theorem 3.14.** If A is an i-v fuzzy KUS-ideal of X, then the set \(X_{\tilde{\mu}_A} = \{x \in X \mid \tilde{\mu}_A (x) = \tilde{\mu}_A (0)\}\) is a KUS-ideal of X.

**Proof.** Let \((z \ast y), (y \ast x) \in X_{\tilde{\mu}_A}\). Then \(\tilde{\mu}_A (z \ast y) = \tilde{\mu}_A (0) = \tilde{\mu}_A (y \ast x)\), and so

\[ \tilde{\mu}_A (z \ast x) \geq r \min \{ \tilde{\mu}_A (z \ast y), \tilde{\mu}_A (y \ast x) \} = r \min \{ \tilde{\mu}_A (0), \tilde{\mu}_A (0) \} = \tilde{\mu}_A (0). \]

Combining this with condition (1) of definition (3.6), we get \(\tilde{\mu}_A (z \ast x) = \tilde{\mu}_A (0)\), that is \((z \ast x) \in X_{\tilde{\mu}_A}\).

Hence \(X_{\tilde{\mu}_A}\) is a KUS-ideal of X. □

IV. Homomorphism of KUS-algebra

**Definition 4.1 (11).** Let \(f : (X; *, 0) \rightarrow (Y; *, 0)\) be a mapping from set X into a set Y. let B be an i-v fuzzy subset in Y. Then the inverse image of B, denoted by \(f^{-1} (B)\), is an i-v fuzzy subset in X with the membership function given by

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\[ \mu_{\Gamma^{-1}(B)}(x) = \mu_B(f(x)), \text{ for all } x \in X. \]

**Proposition 4.2** ([1]). Let \( f \) be a mapping from set \( X \) into a set \( Y \), let \( m = [m^L, m^U] \), and \( n = [n^L, n^U] \) be \( i-v \) fuzzy subsets in \( X \) and \( Y \) respectively. Then

\( f^{-1}(n) = [f^{-1}(n^L), f^{-1}(n^U)] \).

\( f^{-1}(m) = [f^{-1}(m^L), f^{-1}(m^U)] \).

**Theorem 4.3.** Let \( f \) be homomorphism from a KUS-algebra \( X \) into a KUS-algebra \( Y \). If \( B \) is an \( i-v \) fuzzy KUS-ideal of \( Y \), then the inverse image \( f^{-1}(B) \) of \( B \) is an \( i-v \) fuzzy KUS-ideal of \( X \).

**Proof.** Since \( B = [\mu_B^L, \mu_B^U] \) is an \( i-v \) fuzzy KUS-ideal of \( Y \), it follows that from theorem (3.8), that \( (\mu_B^L) \) and \( (\mu_B^U) \) are fuzzy KUS-ideals of \( Y \). Using theorem (2.12), we know \( f^{-1}(\mu_B^L) \) and \( f^{-1}(\mu_B^U) \) are fuzzy KUS-ideals of \( X \). Hence by proposition (4.2), we conclude that \( f^{-1}(B) = [f^{-1}(\mu_B^L), f^{-1}(\mu_B^U)] \) is an \( i-v \) fuzzy KUS-ideal of \( X \).

**Definition 4.4** ([9]). Let \( f \) be a mapping from a set \( X \) into a set \( Y \). Let \( A \) be an \( i-v \) fuzzy set in \( X \). Then the image of \( A \), denoted by \( f(A) \), is the \( i-v \) fuzzy subset in \( Y \) with membership function denoted by :

\[ \mu_{f(A)}(x) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu_A(z) & \text{if } f^{-1}(y) \neq \emptyset, y \in Y \\ [0,0] & \text{otherwise} \end{cases} \]

where \( f^{-1}(y) = \{x \in X | f(x) = y\} \).

**Theorem 4.5.** Let \( f \) be a homomorphism from a KUS-algebra \( X \) into a KUS-algebra \( Y \). If \( A \) is an \( i-v \) fuzzy KUS-ideal of \( X \), then \( f(A) \) is an \( i-v \) fuzzy KUS-ideal of \( Y \).

**Proof.** Assume that \( A = [\mu_A^L, \mu_A^U] \) is an \( i-v \) fuzzy KUS-ideal of \( X \). It follows that from theorem (3.8), that \( (\mu_A^L) \) and \( (\mu_A^U) \) are fuzzy KUS-ideals of \( X \). Using theorem (2.13), that the images \( f(\mu_A^L) \) and \( f(\mu_A^U) \) are fuzzy KUS-ideals of \( Y \). Hence by proposition (4.2), we conclude that \( f(A) = [f(\mu_A^L), f(\mu_A^U)] \) is an \( i-v \) fuzzy KUS-ideal of \( Y \).

**References**