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General Common Fixed Point Theorems for Occasionally Weakly Compatible Mappings

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***Abstract**: The Purpose of this paper is to prove common fixed point theorems in metric space by using occasionally weakly compatible six self mappings satisfying the implicit relation which unify and generalize most of the existing relevant fixed point theorems.*
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I. Introduction

The concept of the commutativity has been generalized in several ways. For this, Sessa, S. [17] has introduced the concept of weakly commuting. Obviously two commuting mappings are weekly commuting but not conversely as given in [17]. Gerald Jungck [4] initiated the concept of compatibility.A weakly commuting pair is compatible but not conversely as given in Jungck [4]. In the later years the concept of compatibility is further generalized in many ways. G. Jungck and P.P. Murthy and Y.J. Cho [5] introduced the concept of compatible mappings of type (A) and they gave some examples to show that compatible maps of type (A) need not be compatible mappings.Extending type (A) mappings H.K.Pathak and M.S.Khan [13] introduced the concept of compatible mappings of type (B) and they gave some examples to show that compatible maps of types (B) need not be compatible mappings of type (A). In 1996,H.K.Pathak, Y.J.Cho, S.S. Chang and S.M.Kang[11] introduced the concept of compatible mappings of type(P) and they gave some examples to show that compatible mappings of type(P) need not be compatible mappings, compatible mappings of type(A), compatible mappings of type(B).In 1998, H.K.Pathak, Y.J.Cho, S.M.Kang and B.Madharia [12] introduced another extension of compatible mappings of type (A) in normed spaces called compatible mappings of type (C) and with some examples they compared these mappings with compatible maps.From the propositions given in [4], [5], [11], [12],[13] we observe that the concept of compatible, compatible mappings of type (A), compatible mappings of type (B),compatible mappings of type(P) and compatible mappings of type (C) are equivalent when S and T are continuous. They are independent if the functions are discontinuous. It has been known from the paper of Kannan [8] that there exists maps that have a discontinuity in the domain but which have fixed points. Moreover, the maps involved in every case were continuous at the fixed point. In 1998, Jungck and Rhoades [6] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversely. Recently in 2006 Jungck and Rhoades [7] introduced occasionally weakly compatible maps(owc) which is more general among the commutativity concepts. The main purpose of this paper is to extend the results of [7] and [15].

II. Preliminaries

Definition 2.1. Let S and T be mappings from a metric space (X, d) into itself. The mappings S and T are said to be

(i) Compatible if $\lim_{n\to\infty} d(STx_n , TSx_n) = 0$

(ii) Compatible of type(A) if $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ and

$\lim_{n\to\infty} d(TSx_n , SSx_n) =0$

(iii) Compatible of type(B) if

$$\lim_{n\to\infty} d(STx_n , TTx_n) \le \frac{1}{2} [\lim_{n\to\infty} d(STx_n , St) + \lim_{n\to\infty} d(St , SSx_n)] \quad \text{and}$$

$$\lim_{n\to\infty} d(TSx_n , SSx_n) \le \frac{1}{2} [\lim_{n\to\infty} d(TSx_n , Tt) + \lim_{n\to\infty} d(Tt , TTx_n)]$$

(iv) Compatible of type(P) if $\lim_{n\to\infty} d(SSx_n , TTx_n) =0$

(v) Compatible of type(C) if

$$\lim_{n\to\infty} d(STx_n, TTx_n) \le \frac{1}{3} [\lim_{n\to\infty} d(STx_n, St) + \lim_{n\to\infty} d(St, SSx_n) + \lim_{n\to\infty} d(St, TTx_n)] \text{ and}$$

$$\lim_{n\to\infty} d(TSx_n, SSx_n) \le \frac{1}{3} [\lim_{n\to\infty} d(TSx_n, Tt) + \lim_{n\to\infty} d(Tt, TTx_n) + \lim_{n\to\infty} d(Tt, SSx_n)]$$

when ever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.2. [6]. A pair of maps T and S is called weakly compatible pair if they commute at coincidence points.

Definition 2.3. [7]. Two self maps S and T on a set X are said to be occasionally weakly compatible(owc) if and only if there is a point $x \in X$ which is a coincidence point of S and T at which S and T commute. i.e., there exists a point $x \in X$ such that $Sx=Tx$ and $STx = TSx$.

III. Implicit Relations.

Let $\mathcal{F}$ be the set of all continuous functions $F : R_+^6 \to R$ satisfying the following conditions:

(3.1) F is non-increasing in variables t_5 and t_6.

(3.2) there exists $h \in (0,1)$ such that for $u, v \ge 0$ with

(3.3) $F(u, v, v, u, u + v, 0) \le 0$ or

(3.4) $F(u, v, u, v, 0, u + v) \le 0$ implies $u \le h.v$.

(3.5) $F(u, u, 0,0, u, u) > 0$ for all $u > 0$.

The following examples of such functions F satisfying (3.1), (3.2), (3.3), (3.4) and (3.5) are available in [15] with verifications and other details.

Example 3.6: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = t_1 - k \max\{ t_2, t_3, t_4, \tfrac{1}{2}(t_5 + t_6) \}, \text{ where } k \in (0,1).$$

Example 3.7:. Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = t_1^2 - t_1(\alpha t_2 + \beta t_3 + \gamma t_4) - \eta t_5 t_6,$$

where $\alpha > 0$; $\beta, \gamma, \eta \ge 0$; $\alpha + \beta + \gamma < 1$ and $\alpha + \eta < 1$.

Example 3.8: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$as

$$F(t_1, t_2,\ldots,t_6) = t_1^3 - \alpha t_1^2 t_2 - \beta t_1 t_3 t_4 - \gamma t_5^2 t_6 - \eta t_5 t_6^2,$$

where $\alpha > 0$; $\beta, \gamma, \eta \ge 0$; $\alpha + \beta < 1$ and $\alpha + \gamma + \eta < 1$.

Example 3.9: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = t_1^3 - \alpha \frac{t_3^2 t_4^2 + t_5^2 t_6^2}{1+t_2+t_3+t_4}, \text{ where } \alpha \in (0,1).$$

Example 3.10: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = t_1^2 - \alpha t_2^2 - \beta \frac{t_5 t_6}{1+t_3^2+t_4^2}, \text{ where } \alpha > 0, \beta \ge 0 \text{ and } \alpha + \beta < 1.$$

Example 3.11: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$$F(t_1, t_2,\ldots,t_6) = \begin{cases} t_1 - a_1 \frac{t_3^2 + t_4^2}{t_3+t_4} - a_2 t_2 - a_3 (t_5+t_6), & \text{if } t_3+t_4 \ne 0 \\ t_1, & \text{if } t_3+t_4 = 0 \end{cases}$$

Where $a_i \ge 0$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 < 1$.

(3.1): Obvious.

(3.2)((3.3)): Let $u > 0$, $F(u,v,v, u, u + v, 0) = u - a_1(v^2 + u^2) / (v + u) - a_2 v - a_3(u + v) \le 0$. If $u \ge v$, then $u \le (a_1 + a_2 + 2a_3)u < u$ which is a contradiction. Hence $u < v$ and $u \le hv$ where $h \in (0, 1)$.

(3.4): Similar argument as in (3.3).

(3.5): $F(u, u, 0, 0, u, u) = u > 0$ for all $u > 0$.

We also add the following examples [16] without verification.

Example 3.12: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$as

$$F(t_1, t_2,\ldots,t_6) = \begin{cases} t_1 - \alpha t_2 - \frac{\beta t_3 t_4 + \gamma t_5 t_6}{t_3+t_4}, & \text{if } t_3+t_4 \ne 0 \\ t_1, & \text{if } t_3+t_4 = 0 \end{cases}$$

Where $\alpha, \beta, \gamma \ge 0$ such that $1 < 2\alpha + \beta < 2$.

Example 3.13: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5 - a_5t_6$ where $\sum_{i=1}^{5} a_i < 1$

Example 3.14: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$as

$F(t_1, t_2,\ldots,t_6) = t_1 - \alpha[\beta \max \{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\} +$

$(1-\beta)[\max\{t_2^2, t_3t_4, t_5t_6, \frac{t_3t_6}{2}, \frac{t_4t_5}{2}\}]^{\frac{1}{2}}]$, where $\alpha \in (0, 1)$ and $0 \le \beta \le 1$.

Example 3.15: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1^2 - \alpha \max \{t_2^2, t_3^2, t_4^2\} - \beta \max\{\frac{t_3t_5}{2}, \frac{t_4t_6}{2}\} - \gamma t_5t_6$.

Where $\alpha, \beta, \gamma, \ge 0$ and $\alpha + \beta + \gamma < 1$.

Popa et al[16], noticed that Husain and Sehgal [3] type contraction conditions (e.g. [2,9,10,18]) can be deduced from similar implicit relations in addition to all earlier ones if we slightly modified(3.1) as follows:

(3.1)' F is decreasing in variables $t_2,\ldots, t_6$.

Hereafter, let $F : R_+^6 \to R$be a continuous function which satisfy the conditions (3.1)',(3.2), (3.3), (3.4) and (3.5) and let ψ be the family of such functions F. Some examples of [16].

Example 3.16: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1 - \varphi(\max \{t_2, t_3, t_4, \frac{1}{2}(t_5+t_6)\})$

Where $\varphi : R^+ \to R^+$ is an increasing upper semi continuous function with $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$.

(3.1)': Obvious.

(3.2)((3.3)): Let $u > 0$. $F(u, v, v, u, u + v, 0) = u - \varphi(\max \{v, v, u,(u+v)/2\}) \le 0$.

If $u \ge v$, then $u \le \varphi(u) < u$ which is a contradiction. Hence $u < v$ and $u \le hv$

Where $h \in (0, 1)$.

(3.4): Similar argument as in (3.3).

(3.5): $F(u, u,0,0, u, u) = u - \varphi(\max \{u,0,0,(u+u)/2\}) = u - \varphi(u) > 0$ for all $u > 0$.

Example 3.17: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1 - \varphi(t_2,t_3,\ldots,t_6)$

Where $\varphi : R_+^5 \to R^+$ is an upper semi continuous and non decreasing function in each coordinate variable such that $\varphi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \ge 0$ with

$\alpha + \beta + \gamma \le 3$.

Example 3.18: Define $F(t_1, t_2,\ldots,t_6) : R_+^6 \to R$ as

$F(t_1, t_2,\ldots,t_6) = t_1^2 - \varphi(t_2^2, t_3t_4, t_5t_6, t_3t_6, t_4t_5)$

Where $\varphi : R_+^5 \to R^+$ is an upper semi continuous and non decreasing function in each coordinate variable such that $\varphi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \ge 0$

with $\alpha + \beta + \gamma \le 3$.

Here it may be noticed that all earlier mentioned examples continue to enjoy the format of modified implicit relation as adopted herein.

IV. Main Result

Now we state our first main result:

4.1 Theorem : L, M, A, B, S and T be self mappings of a metric space (X, d) satisfying the conditions

(4.1.1) $F(d(Lx, My), d(ABx, STy), d(ABx, Lx), d(STy, My), d(ABx, My), d(STy, Lx)) \le 0$

(4.1.2) $L(X) \subseteq ST(X)$ and $M(X) \subseteq AB(X)$.

If one of L(X),M(X),AB(X) or ST(X) is a complete subspace of X, then

(4.1.3) the pair (L,AB) has a point of coincidence,

(4.1.4) the pair (M, ST) has a point of coincidence.

Moreover, L,M,AB and ST have a unique common fixed point provided both the pairs (L, AB) or (M, ST) is occasionally weakly compatible mappings.

Further if

(A, B), (S,T), (M,T), (L,T),(M,B) are commuting mappings then A,B,S,T,L and M have a unique common fixed point.

Proof. Since $L(X) \subseteq ST(X)$, for arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Lx_0 = STx_1$. Since $M(X) \subseteq AB(X)$, for the point x_1, we can choose a point $x_2 \in X$ such that $Mx_1 = ABx_2$ and so on. Inductively, we can define a sequence $\langle y_n \rangle$ in X such that

(4.1.5) $y_{2n} = Lx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Mx_{2n+1} = ABx_{2n+2}$; n = 0,1,2, …

From (4.1.1) we have

$F(d(Lx_{2n+2},Mx_{2n+1}),d(ABx_{2n+2},STx_{2n+1}),d(ABx_{2n+2},Lx_{2n+2}),d(STx_{2n+1},Mx_{2n+1}),$
$d(ABx_{2n+2},Mx_{2n+1}), d(STx_{2n+1},Lx_{2n+2})) \le 0$

or $F(d(y_{2n+2}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+1}), d(y_{2n}, y_{2n})) \le 0$

or $F(d(y_{2n+2}, y_{2n+1}), 0, d(y_{2n+2}, y_{2n+1}), 0, 0, 0) \leq 0$
or $F(d(y_{2n+2}, y_{2n+1}), 0, d(y_{2n+2},y_{2n+1}), 0,0,d(y_{2n+2}, y_{2n+1})) \leq 0$
Yielding thereby $d(y_{2n+2}, y_{2n+1}) = 0$ (due to (3.4)). Similarly, using (3.3) we can show that $d(y_{2n+1}, y_{2n}) = 0$. Thus it follows that $d(y_n, y_{n+1}) = 0$ for every $n \in N$.

Let us write $d_n = d(y_n, y_{n+1})$, n = 0,1, 2,…. First we shall prove that $<d_n>$is a non-decreasing sequence in R^+.
From (4.1.1), we have

$F(d(Lx_{2n},Mx_{2n+1}), d(ABx_{2n}, STx_{2n+1}), d(ABx_{2n}, Lx_{2n}), d(STx_{2n+1},Mx_{2n+1}),$
$d(ABx_{2n},Mx_{2n+1}), d(STx_{2n+1}, Lx_{2n})) \leq 0,$
or $F(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),d(y_{2n-1}, y_{2n+1}), d(y_{2n}, y_{2n})) \leq 0,$
or $F(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),$
$d(y_{2n-1}, y_{2n+1}) + d(y_{2n-1}, y_{2n}) + d(y_{2n+1}, y_{2n}), 0) \leq 0$
or $F(d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1}),d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}), 0) \leq 0$
Implying thereby $d_{2n} \leq hd_{2n-1}< d_{2n-1}$ (due to (3.3)). Similarly using (3.4), we have $d_{2n+1}\leq hd_{2n}$. Thus $d_{n+1} < d_n$ for n = 0, 1, 2, … Now proceeding on the lines of the proof of Lemma 3.2 [14, p.355], we can show that $d(y_i, y_j) = 0$ for $i, j \in N$.
Now we show$<y_n>$is a sequence in a metric space (X, d) described by (4.1.5), then

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0.$

We have $d_{2n+1}\leq hd_{2n}$ and $d_{2n} \leq hd_{2n-1}$. Therefore, we obtain $d_n \leq h^n d_0$. Hence $\lim_{n\to\infty} d(y_n, y_{n+1}) = \lim_{n\to\infty} d_n= 0$.

The sequence $<y_n>$described by (4.1.5) is a Cauchy sequence.
Since $\lim d (y_n, y_{n+1}) = 0$,it is sufficient to show that a subsequence $<y_{2n}>$of $<y_n>$is a Cauchy sequence in X.
Suppose that $<y_{2n}>$is not a Cauchy sequence in X. Then for every $\in>0$ there exists strictly increasing sequences $<m_k>$, $<n_k>$of positive integers such that $k \leq n_k< m_k$ with $d(y_{2n_k-1},y_{2m_k}) \geq \in$ and $d(y_{2n_k} , y_{2m_k-2}) <\in$. Now proceeding on the lines of the proof of Lemma 1.3[1] (or Lemma 3.3[14]), we obtain

$\lim_{n\to\infty} d(y_{2n_k}, y_{2m_k}) = \in, \lim_{n\to\infty} d(y_{2n_k}, y_{2m_k-1}) = \in, \lim_{n\to\infty} d(y_{2n_k+1},y_{2m_k}) = \in$ and

$\lim_{n\to\infty} d(y_{2n_k+1}, y_{2m_k-1}) = \in$. Now using (4.1.1), we have

$F(d(Lx_{2m_k},Mx_{2n_k+1}), d(ABx_{2m_k} , STx_{2n_k+1}), d(ABx_{2m_k} ,Lx_{2m_k}),$
$d(Mx_{2n_k+1}, STx_{2n_k+1}), d(ABx_{2m_k} ,Mx_{2n_k+1}), d(STx_{2n_k+1},Lx_{2m_k}) \leq 0$
or$F(d(y_{2m_k} , y_{2n_k+1}), d(y_{2m_k-1},y_{2n_k}), d(y_{2m_k-1}, y_{2m_k}),$
$d(y_{2n_k} , y_{2n_k+1}), d(y_{2m_k-1}, y_{2n_k+1}), d(y_{2n_k} , y_{2m_k})) \leq 0.$
Letting $n \to \infty$, we have $F(\in,\in,0,0,\in,\in) \leq 0$, which is a contradiction to (3.5).
Therefore $<y_{2n}>$ is a Cauchy sequence.
Suppose that AB(X) is a complete subspace of X then the subsequence $\{y_{2n+1}\}$ which is contain in AB(X). We must have a limit z in AB(X).

As $<y_n>$is a Cauchy sequence containing a convergent subsequence $<y_{2n+1}>$, therefore $<y_n>$also converges implying thereby the convergence of the subsequence $<y_{2n}>$, i.e. $\lim Lx_{2n}= \lim Mx_{2n+1} = \lim STx_{2n+1} = \lim ABx_{2n+2} = z$. Let $u \in (AB)^{-1}(z)$, then
$ABu= z$. If $Lu \neq z$, then using (4.1.1), we have
$F(d(Lu,Mx_{2n-1}), d(ABu, STx_{2n-1}), d(ABu, Lu), d(STx_{2n-1},Mx_{2n-1}),$
$d(ABu,Mx_{2n-1}), d(STx_{2n-1}, Lu)) \leq 0$
which on letting $n \to \infty$, reduces to

$F(d(Lu, z), d(z, z), d(z, Lu); d(z, z), d(z, z), d(z, Lu)) \leq 0$
or $F(d(Lu, z), 0, d(z, Lu), 0, 0, d(z, Lu)) \leq 0$
implying thereby $d(z, Lu) = 0$ (due to (3.4)).
Hence $z = Lu =ABu$.
Since $L(X) \subseteq ST(X)$, there exists $v \in(ST)^{-1} (z)$ such that $STv= z$. By (4.1.1), we have

$F(d(Lu,Mv), d(ABu, STv), d(ABu, Lu), d(STv, Mv), d(ABu,Mv), d(STv, Lu)) \leq 0$
or $F(d(z,Mv), 0, 0, d(z,Mv), d(z,Mv), 0) \leq 0$
yielding thereby $d(z,Mv) = 0$ (due to (3.3)). Therefore
$z = Mv$. Hence $Lu = ABu= Mv= STv= z$ which establishes (4.1.3) and (4.1.4).
If one assumes that ST(X) is a complete subspace of X, then analogous arguments establish (4.1.3) and (4.1.4).
The remaining two cases also pertain essentially to the previous cases. Indeed, if L(X) is complete, then $z \in L(X) \subseteq ST(X)$. Similarly if M(X) is complete, then $z \in M(X) \subseteq S(X)$. Thus in all cases, (4.1.3) and (4.1.4) are completely established.

Suppose L and AB are occasionally weakly compatible.
By occasionally weakly compatible of L, AB gives Lu = ABu and L(AB)u = (AB)Lu.
Occasionally weakly compatible of M, ST gives Mv = STv and M(ST)v = (ST)Mv.
L(AB)u = (AB)Lu and LLu = L(AB)u = (AB)Lu = (AB)(AB)u.
We have MMv = M(ST)v = (ST)Mv = (ST)(ST)v.
Form (4.1.1) $F(d(LLu,Mv), d((AB)Lu, STv), d((AB)Lu, LLu),$
$d(STv,Mv), d((AB)Lu,Mv), d(STv; LLu)) \le 0$
or $F(d(LLu,Mv), d(LLu,Mv), 0, 0, d(LLu,Mv),d(LLu,Mv)) \le 0$.
Contradition to (3.5) if d(LLu,Mv)> 0.
Hence LLu = Mv = Lu. So Lu = LLu = (AB)Lu.
Therefore Lu is a common fixed point of L and AB.
Similarlly we can prove that Mv (=Lu) is a common fixed point of M and ST.

To Prove Uniqueness of Lu
Suppose that Lu andLw, Lu = Lw are common fixed point of L, M, ST and AB by (4.1.1)
$F(d(LLu,MLw), d((AB)Lu, (ST)Lw), d(LLu, ABLu),$
$d(MLw,STLw), d(LLu,(ST)Lw), d(MLw, ABLu)) \le 0$
or $F(d(Lu,Lw, a), d(Lu,Lw, a)), 0, 0, d(Lu,Lw, a),d(Lu,Lw, a)) \le 0$.
Which shows Lu = Lw. But Lu = z, so z is the common fixed point of L, M, ST and AB.

Finally we need to show that z is a common fixed point of A,B,L,M,S and T.
Since (A,B), (A,L), (B,L) are commutative
Az = A(ABz) = A(BAz) = (AB)(Az); Az = ALz = LAz
Bz = B(ABz) = (BA)(Bz) =(AB)(Bz); Bz = BLz = LBz.
Which shows that Az, Bz are common fixed points of (AB, L) yielding there by Az = z = Bz = Lz = ABz in the view of uniqueness of common fixed point of the pairs (AB, L).
Similarly using the, commutativity of (S, T), (S, M) and (T, M) it can be shown that
Sz = z = Tz = Mz = STz. Now, we need to show that Az = Sz (Bz = Tz) also remains a common fixed point of both the pairs (AB, L) and (ST, M) from 4.1.1 we can easily prove Az = Sz and Bz = Tz. Which shows z is a common fixed point of A, B, L, M, S and T.

4.1.6 Corollary : The conclusions of Theorem 4.1 remain true if (for all x, y $\in$ X) implicit relation (4.1.1) is replaced by any one of the following.

(4.1.7) $d(Lx, My) \le k \max\{ d(ABx, STy), d(ABx, Lx), d(STy,My), \frac{1}{2}(d(ABx,My) + d(STy, Lx))\}$, where $k \in (0,1)$.

(4.1.8) $d^2(Lx,My) \le d(Lx,My)[\alpha d(ABx, STy)+ \beta d(ABx, Lx)+ \gamma d(STy,My)] + \eta d(ABx,My).d(STy; Lx)$
Where $\alpha >0$; $\beta,\gamma,\eta \ge 0$, $\alpha + \beta + \gamma <1$ and $\alpha + \eta <1$.

(4.1.9) $d^3(Lx,My) \le \alpha d^2(Lx,My)d(ABx, STy) + \beta d(Lx,My)d(ABx, Lx)d(STy,My) + \gamma d^2(ABx,My)d(STy,Lx) + \eta d(ABx,My)d^2(STy, Lx)$
Where $\alpha >0$; $\beta,\gamma,\eta \ge 0$, $\alpha + \beta <1$ and $\alpha + \eta + \gamma <1$.

(4.1.10) $d^3(Lx,My) \le \alpha \frac{d^2(ABx,Lx)d^2(STy,My)+ d^2(ABx,Ly)d^2(STy,Lx)}{1+d(ABx,STy)+ d(ABx,Lx)+ d(STy,My)}$
Where $\alpha \in (0, 1)$.

(4.1.11) $d^2(Lx,My) \le \alpha d^2(ABx, STy) + \beta \frac{d(ABx,My)d(STy,Lx)}{1+d^2(ABx,Lx)+ d^2(STy,My)}$
Where $\alpha >0$, $\beta \ge 0$ and $\alpha + \beta <1$.

(4.1.12) $d(Lx,My) \le a_1\frac{d^2(ABx,Lx)+ d^2(STy,My)}{d(ABx,Lx)+d(STy,My)} + a_2 d(ABx, STy) + a_3(d(ABx, My) + d(STy, Lx))$
Where $a_i \ge 0$ with at least one a_i non zero and $a_1 + a_2 + 2a_3 <1$.

(4.1.13) $d (Lx,My) \le \alpha d(ABx, STy) + \frac{\beta d (ABx,Lx)d(STy,My)+ \gamma d(ABx,My)d(STy,Lx)}{d(ABx,Lx)+ d(STy,My)}$
Where $\alpha, \beta, \gamma \ge 0$ such that $1 <2\alpha + \beta <2$.

(4.1.14)) $d(Lx,My) \le a_1d(ABx, STy) + a_2d(ABx, Lx)+ a_3d(STy,My) + a_4d(ABx,My) + a_5d(STy, Lx)$, where $\sum_{i=1}^{5} a_i < 1$

(4.1.15) $d(Lx,My) \le \alpha[\beta \max \{d(ABx, STy), d(ABx, Lx), d(STy,My), \frac{1}{2} (d(ABx,My) + d(STy, Lx)) \}+ (1- \beta)$

[max d^2(ABx, STy), d(ABx, Lx)d(STy,My), d(ABx,My)d(STy, Lx), $\frac{d(ABx,Lx)d(STy,Lx)}{2}$, $\frac{d(STy,My)d(ABx,My)}{2}$}]$^{\frac{1}{2}}$]

Where $\alpha \in (0, 1)$ and $0 \le \beta \le 1$.

(4.1.16) $d^2(Lx,My) \le \alpha \max \{d^2(ABx,STy), d^2(ABx, Lx), d^2(STy,My)\}$
$+\beta\max\{\frac{d(ABx,Lx)d(ABx,My)}{2}, \frac{d(STy,My)d(STy,Lx)}{2}\}$
$+ \gamma d(ABx,My)d(STy, Lx)$

Where $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma < 1$.

(4.1.17) $d(Lx,My) \le \varphi (\max\{d(ABx, STy), d(ABx, Lx), d(STy,My), \frac{1}{2}[d(ABx,My) + d(STy,Lx)]\})$

Where $\varphi : R^+ \rightarrow R^+$ is an upper semicontinuous and increasing function with $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$.

(4.1.18) $d(Lx,My) \le \varphi (d(ABx, STy), d(ABx, Lx), d(STy,By), d(ABx,My), d(STy,Lx))$

where $\varphi : R_+^5 \rightarrow R^+$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that $\varphi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma \le 3$

(4.1.19) $d^2(Lx,My) \le \varphi (d^2(ABx, STy), d(ABx, Lx)d(STy,My), d(ABx,My)d(STy,Lx), d(ABx, Lx)d(STy, Lx), d(STy, My)d(ABx,My))$

where $\varphi : R_+^5 \rightarrow R^+$ is an upper semicontinuous and nondecreasing function in each coordinate variable such that $\varphi(t, t, \alpha t, \beta t, \gamma t) < t$ for each $t > 0$ and $\alpha, \beta, \gamma \ge 0$ with $\alpha + \beta + \gamma \le 3$

Proof. The proof follows from Theorem 4.1 and Examples 3.6 to 3.18 the above.

Put B = T = I_x in corollary 4.1.6, we get result in four self maps.
If we put B = T= I_x where I_x identity self map on X in theorem 4.1 then we get the following result.

4.1.20 **Corollary**: L,M,A and S be self mappings of a metric space (X, d) satisfying the conditions
$F(d(Lx,My), d(Ax, Sy), d(Ax, Lx), d(Sy,My), d(Ax,My), d(Sy, Lx)) \le 0$
for all $x,y \in X$ where F enjoys the property (3.5) and $L(X) \subseteq S(X)$ and $M(X) \subseteq A(X)$.
If one of L(X), M(X),A (X) or S (X) is a complete subspace of X, then
(4.1.21) the pair (L,A) has a point of coincidence,
(4.1.22) the pair (M, S) has a point of coincidence.
Moreover, L,M,A and Shave a unique common fixed point provided both the pairs (L, A) or (M, S) is occasionally weakly compatible mappings.

Taking L= M and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.23 **Corollary**: L,A and S be self mappings of a metric space (X, d) satisfying the conditions
$F(d(Lx,Ly), d(Ax, Sy), d(Ax, Lx), d(Sy,Ly), d(Ax,Ly), d(Sy, Lx)) \le 0$ for all $x,y \in X$ where F enjoys the property (3.5) and $L(X) \subseteq S(X)$ and
$L(X) \subseteq A(X)$.. If one of L(X),A (X) or S (X) is a complete subspace of X, then
(4.1.24) the pair (L,A) has a point of coincidence,
(4.1.25) the pair (L, S) has a point of coincidence.
Moreover, L, A and Shave a unique common fixed point provided both the pairs (L, A) or
(L, S) is occasionally weakly compatible mappings.

Taking A= S and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.26 **Corollary**: L,M and S be self mappings of a metric space (X, d) satisfying the conditions
$F(d(Lx,My), d(Sx, Sy), d(Ax, Lx), d(Sy,My), d(Sx,My), d(Sy, Lx)) \le 0$
for all $x,y \in X$ where F enjoys the property (3.5) and $L(X) \subseteq S(X)$ and $M(X) \subseteq S(X)$.
If one of L(X),M(X) or S (X) is a complete subspace of X, then
(4.1.27) the pair (L,S) has a point of coincidence,
(4.1.28) the pair (M, S) has a point of coincidence.
Moreover, L,M and Shave a unique common fixed point provided both the pairs (L, S) or (M,S) is occasionally weakly compatible mappings.

Taking L= M, A = S and T = B = I_X in theorem 4.1 we get the following corollary.

4.1.29 **Corollary**: L and A be self mappings of a metric space (X, d) satisfying the conditions
$F(d(Lx,Ly), d(Ax, Ay), d(Ax, Lx), d(Ay,Ly), d(Ax,Ly), d(Ay, Lx)) \leq 0$
for all $x,y \in X$ where F enjoys the property (3.5) and $L(X) \subseteq A(X)$. If one of L(X) or A (X) is a complete subspace of X, then
(4.1.30) the pair (L,A) has a point of coincidence,
Moreover, L and Ahave a unique common fixed point provided the pair (L, A) is occasionally weakly compatible mappings.

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