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On Quasi h^*-open and Quasi h^*-closed Functions

J.Antony Rex Rodrigo and A.Hana Selvi Jansi
Department Of Mathematics, V.O.Chidambaram College, Tuticorin-628 008.
Department Of Mathematics, Sarah Tucker College, Tirunelveli -627 007.

Abstract : *The purpose of this paper is to give a new type of open function called quasi h^*-open function. Also, we obtain its characterizations and its basic properties.*
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I. Introduction

1. Introduction and Preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science .Many different forms of open functions have been introduced over the years .Recently, as a generalization of closed sets, the notion of h^*-closed sets were introduced by J. Antony Rex et al [2].In this paper,we will continue the study of related functions by involving h^*-open set. We introduce and charaterzie the concept of h^*- open functions.

Throughout this paper (X, τ),(Y, σ) and (Z, η) represent topological spaces on which no separation axiom is defined unless otherwise mentioned .For a subset A of a space (X ,τ),cl (A) and int (A) denote the closure of A and the interior of A in X respectively .

DEFINITION 1.1: *A subset A of a space (X ,τ) is called* a semi-open set [9] if A $\subset$ cl(int(A)) and a semi closed set if int(cl(A)) $\subset$ A.

DEFINITION 1.2 : *A subset A of a space (X ,τ) is called a*

(1) ω-closed set [11] if cl(A) $\subset$ U whenever A $\subset$ U and U is semi-open in X. The complement of a ω-closed set is called a ω-open set.

(2) h^*-closed set [2] if scl(A) $\subset$ int (U) whenever A $\subset$ U and U is ω-open in X. The complement of a h^*-closed set is called a h^*-open set.

The union (resp.intersection) of all h^*-open (resp. h^*-closed) sets, each contained in (resp. containing) a set A in a topological space (X, τ) is called the h^*- interior (resp. h^*-closure) of A and is denoted by h^*-int(A) (resp. h^*-cl(A)) [2].

DEFINITION 1.3 : A function f : (X, τ) → (Y,σ) is called

1.a h^* –irresolute [2] (h^* –continuous[1]) if f^{-1} (V) is h^* – open set of (X, τ) for each h^* – open (resp.open) set V of (Y,σ).

2. a h^* –open [2] (resp. h^* –closed[1]) if f (V) is h^* – open (resp. h^* – closed) in (Y,σ) for every open (resp. closed) subset of (X,τ).

II. Quasi h^* –open Functions

DEFINITION 2.1 : A function f : (X, τ) → (Y,σ) is said to be quasi h^* –open if the image of every h^* – open set in X is open in Y.

It is evident that, the concepts quasi h^* –openness and h^* –continuity coincide if the function is a bijection.

THEOREM 2.2 : A function f : (X, τ) → (Y,σ) is quasi h^* –open if and only if for every subset U of X, f(h^*-int(U))$\subset$ int(f(U)).Assume that h^*C(X,τ) is closed under any union.

Proof: Let f be a quasi h^* –open function. Now we have int(U) ⊂ U and h^*-int(U) is a h^* –open set. Hence We obtain that f(h^*-int(U))⊂ f(U).As f(h^*-int(U)) is open, f(h^*-int(U))⊂ int(f(U)).

Conversely, assume that U is a h^* –open set in X. Then,f(U) = f(h^*-int(U)) ⊂ int(f(U)) but int(f(U))⊂ f(U). Consequently, f(U) = int(f(U)) and hence f is quasi h^* –open.

LEMMA 2.3 : If a function f : (X, τ) → (Y,σ) is quasi h^* –open, then is h^*-int (f^{-1}(G))⊂ f^{-1}(int(G)) for every subset G of Y. Assume that $h^*C(X,\tau)$ is closed under any union.

Proof: Let G be any arbitrary subset of Y. Then, h^*-int (f^{-1}(G)) is a h^* –open set in X and f is qusai h^* –open, then f(h^*-int (f^{-1}(G))) ⊂ int (f(f^{-1}(G))) ⊂ int(G). Thus h^*-int (f^{-1}(G)) ⊂ f^{-1}(int(G)).

DEFINITION 2.4 : A subset S is called a h^*- neighbourhood [ ] of a point x of X if there exists a h^* –open set U such that x ∈ U ⊂ S.

THEOREM 2.5 : Assume that $h^*C(X,\tau)$ is closed under any union. For a function f : (X, τ) → (Y,σ) , the following are equivalent:

(1).f is quasi h^* –open;

(2).For each subset U of X, f(h^*-int(U))⊂ int(f(U)).

(3).For each x ∈ X and each h^*- neighbourhood U of x in X, there exists a neighbourhood V of f(x) in Y such that V ⊂ f(U).

Proof: (1) ⇒ (2): It follows from Theorem2.2.

(2) ⇒ (3):Let x ∈ X and U be an arbitrary h^*- neighbourhood of x in X. Then there exists a h^* –open set W in X such that x ∈ W ⊂ U.Then by (2), we have f (W) = f(h^*-int(W))⊂ int(f(W)) and hence f(W) = int (f(W)). Therefore it follows that f(W) is open in Y such that f(x) ∈ f(W) ⊂ f(U).BY taking V=f(W) we get the result.

(3) ⇒ (1): Let U be an arbitrary h^* – open set in X. Then for each y ∈ f(U),by (3) there exists a neighbourhood V_y of y in Y such that V_y ⊂ f(U).As V_y is a neighbourhood of y, there exists an open set W_y in Y such that y ∈ W_y ⊂ V_y.Thus f(U) = ∪{ W_y: y ∈ f(U)} which is an open set in Y.This implies that f is a quasi h^* –open function.

THEOREM 2.6 : A function f : (X, τ) → (Y,σ) is quasi h^* –open if and only if for any subset B of Y and for any h^* – closed set F of X containing f^{-1}(B), there exists a closed set G of Y containing B such that f^{-1}(G) ⊂ F.

Proof: Suppose f : (X, τ) → (Y,σ) is quasi h^* –open. Let B ⊂ Y and F be a h^* – closed set of X containing f^{-1}(B). Now put G = Y – f(X−F). It is clear that f^{-1}(B) ⊂ F implies B ⊂ G. Since f is quasi h^* –open, we obtain G as a closed set of Y. Moreover, we have f^{-1}(G) ⊂ F.

Conversely, let U be a h^* –open set of X and put B = Y –f(U). Then X – U is a h^* – closed set in X containing f^{-1}(B). By hypothesis, there exists a closed set F of Y such that B ⊂ F and f^{-1}(F) ⊂ X−U.Hence, we obtain f(U) ⊂ Y−F. On the other hand, it follows that B ⊂ F, Y− F⊂ Y−B = f(U). Thus, we obtain f(U) = Y−F which is open and hence f is a h^* –open function.

THEOREM 2.7 : A function f : (X, τ) → (Y,σ) is quasi h^* –open if and only if f^{-1}(cl(B)) ⊂ h^*- cl (f^{-1}(B)) for every subset B of Y.

Proof : Suppose that f is quasi h^* –open . For any subset B of Y, f^{-1}(B) ⊂ h^*- cl (f^{-1}(B)).Therefore by Theorem 2.6, there exists a closed set F in Y such that B ⊂ F and f^{-1}(F) ⊂h^*- cl (f^{-1}(B)).Therefore we obtain f^{-1}(cl(B)) ⊂ h^*- cl(f^{-1}(B)).

Conversely, let B ⊂ Y and F be a h^* – closed set of X containing f^{-1}(B). Put W = cl_Y (B), then we have B ⊂ W and W is closed and f^{-1}(W) ⊂ h^*- cl(f^{-1}(B)) ⊂ F. Then by Theorem 2.6, f is a quasi h^* –open function.

LEMMA 2.8 : Let f : (X, τ) → (Y,σ) and g: (Y,σ) →(Z,η) be two functions and g∘f : (X, τ) → (Z,η) is quasi h^* –open. If g is continuous injective, then f is quasi h^* –open.

Proof: Let U be a h^* –open set in X ,then (g∘f) (U) is open in Z since g∘f is quasi h^* –open. Again g is an injective continuous function, f(U) = g^{-1}(g∘f (U)) is open in Y. This shows that f is quasi h^* –open

III. Quasi h^* –closed Functions

DEFINITION 3.1 : A function f : (X, τ) → (Y, σ) is said to be quasi h^* – closed if the image of every h^* – closed set in X is closed in Y.

Clearly every quasi h^* – closed function is closed as well as h^* – closed.

REMARK 3.2 : Every h^* – closed (resp. closed) function need not be a quasi h^* – closed function as shown by the following example.

EXAMPLE 3.3 : Let X = Y = {a ,b,c},τ = {φ,{a},{a,b},X} and σ = {φ, {a},{a,b},{a,c},Y } . Here h^*C(X,τ) = {φ,{b},{c},{b,c},{a,c},X},h^*C(Y,σ) = {φ,{b},{c},{b,c},Y}.Then the identy function f : (X, τ) → (Y,σ) is h^* – closed as well as closed but not quasi h^* – closed.

LEMMA 3.4 : If a function f : (X, τ) → (Y, σ) is quasi h^* – closed, then (f^{-1}(int (B)))⊂ h^*-int (f^{-1}(B)) for every subset B of Y.

Proof : This proof is similar to the proof of Lemma 2.3.

THEOREM 3.5 : A function f : (X, τ) → (Y,σ) is quasi h^* – closed if and only if for any subset B of Y and for any h^* – open set G of X containing f^{-1}(B), there exists an open set U of Y containing B such that f^{-1}(U) ⊂ G.

Proof: This proof is similar to the proof of Theorem 2.6.

DEFINITION 3.6 : [1] A function f : (X, τ) → (Y,σ) is called strongly h^* – closed if the image of every h^* – closed subset of X is h^* – closed in Y.

THEOREM 3.7 : If f : (X, τ) → (Y ,σ) and g: (Y, σ) →(Z ,η) are two quasi h^* – closed functions, then g∘f : (X, τ) → (Z ,η) is a quasi h^* –open function.

Proof: Obvious.

THEOREM 3.8: Let f : (X, τ) → (Y,σ) and g: (Y,σ) →(Z,η) be any two functions. Then: (1) If f is h^* – closed and g is quasi h^* – closed, then g∘f is closed;

(2) If f is quasi h^* – closed and g is h^* – closed, then g∘f is strongly h^*– closed;

(3) If f is strongly h^* – closed and g is quasi h^* – closed, then g∘f is quasi h^* – closed.

Proof: Obvious.

THEOREM 3.9 : Let f : (X,τ) → (Y,σ) and g: (Y,σ) →(Z,η) be any two functions such that g∘f : (X, τ) → (Z,η) is quasi h^* – closed.

(1).If f is h^* – irresolute surjective , then g is closed.

(2).If g is h^* – continuous injective, then f is strongly h^* – closed.

Proof: (1). Suppose that F is an arbitrary closed set in Y, then it is a h^* – closed set in Y. As f is h^* – irresolute, f^{-1}(F) is h^* – closed in X .Since g∘f : (X, τ) → (Z,η) is quasi h^* – closed and f is surjective .(g∘f)(f^{-1}(F)) = g(F), which is closed in Z. This implies that g is a closed function.

(2).Suppose F is any h^* – closed in X. Since g∘f : (X, τ) → (Z,η) is quasi h^* – closed , (g∘f) (F) is closed in Z. Again g is a h^* – continuous injective function, g^{-1}(g∘f(F)) = f (F), which is h^* – closed in Y. This shows that f is strongly h^* – closed.

THEOREM 3.10 : Let (X, τ) and (Y,σ) be topological spaces. Then the function g : (X, τ) → (Y,σ) is quasi h^* – closed if and only if g(X) is closed in Y and g(V)−g(X – V) is open in g(X) whenever V is h^* –open in X.

Proof: Necessity: Suppose g: (X, τ) → (Y, σ) is a quasi h^* – closed function. Since X is h^* – closed, g(X) is closed in Y and g(V) \ g(X\ V) = g(V)∩(g(X) \g(X\ V)) is open in g(X) when V is h^* –open in X.

Sufficiency: Suppose g(X) is closed in Y, g(V) \ g(X\ V) is open in g(X) when V is h^* –open in X, and let C be closed in X .Then g(C) = g(X) \ (g(X\ C) \g(C)) is closed and hence, closed in Y.

COROLLARY 3.11: Let (X, τ) and (Y,σ) be topological spaces. Then a surjective function g : (X, τ) → (Y,σ) is quasi h^* – closed if and only if g(V) \ g(X\ V) is open in Y whenever V is h^* –open in X.

Proof: Obvious.

COROLLARY 3.12: Let (X, τ) and (Y,σ) be topological spaces and let g : (X, τ) → (Y,σ) be a quasi h^* – closed surjective function. Then the topology on Y is {g(V) \ g(X\ V): V is h^* –open in X}.

Proof: Let W be open in Y.Then g^{-1}(W) is h^* – open in X, and g(g^{-1}(W)) \ g(X \ g^{-1}(W)) = W. Hence, all open sets in Y are of the form g(V) \ g(X\ V), V is h^* –open in X. On the other hand, all sets of the form g(V) \ g(X\ V), V is h^* –open in X, are open in Y from corollary 3.11.

DEFINITION 3.13: A topological space (X, τ) is said to be strongly h^* - normal if for any pair of disjoint h^* – closed subsets F_1 and F_2 of X, there exist disjoint h^* - open sets U and V such that F_1⊂ U and F_2⊂ V.

THEOREM 3.14 : Let (X, τ) and (Y,σ) be topological spaces with X is strongly h^* - normal . If g : (X, τ) → (Y,σ) is a h^* – continuous , quasi h^* – closed surjective function. Then Y is normal.

Proof: Let K and M be disjoint closed subsets of Y. Then g^{-1}(K), g^{-1}(M) are disjoint h^* – closed subsets of X . Since X is strongly h^* - normal, there exist disjoint h^* - open sets V and W such that g^{-1}(K) ⊂ V and g^{-1}(M) ⊂ W . Then K ⊂ g(V) –g (X– V) and M ⊂ g(W) –g(X– W).Further by corollary 3.11, g(V)−g(X– V) and g(W) –g(X– W) are open sets in Y and clearly g(V)−g(X– V)∩ g(W) –g(X– W) = φ. This shows that Y is normal.

REFERENCES

[1] J. Antony Rex Rodrigo and A. Hana Selvi Jansi,Topological Mappings via h^* - sets,International Journal of Science,Engineering and Technology Research (IJSETR) Volume 2,Issue 1, January 2013.

[2] J. Antony Rex Rodrigo and A. Hana Selvi Jansi, On h^* - closed sets in topological spaces.(communicated)

[3] J. Antony Rex Rodrigo, A. Hana Selvi Jansi, and Jessie Theodore - On h^* - open sets in topological spaces.(communicated)

[4] K. Balachandran, P. Sundaram and H. Maki. On generalized continuous maps in topological spaces . Mem, Fac. Sci. Kochi Univ. Ser. A. Math. (1991), 5 – 13.

[5] P. Bhattacharya and B. K. Lahiri, Semi – generalized closed sets in topology, Indian J . Pure. Appl, Math., closed sets in topology, (1987), 375 – 382.

[6] S.G. Crossley and S. K. Hildebrand Semi – topological properties, Fund. Math., (1972). 233 – 254.

[7] R. Devi, H. Maki and K. Balachandran Semi – generalized homeomorphisms and generalized Semi - homeomorphisms in topological spaces, Indian J . Pure. Appl, Math., (1995), 271 – 284.

[8] N. Levine.Generalized closed sets in topology, Rend. Circ. Math.. Palermo, (1970), 89 – 96.

[9] N. Levine, Semi – open sets and semi – continuity in topological spaces, Amer.Math.Monthly, (1963), 36-41.

[10] N. Rajesh and E. Ekici , On Quasi $\widetilde{gs}$ - open and Quasi $\widetilde{gs}$ - closed Functions , Bull. Malays. Math.Sci.Soc.(2)31(20(2008),217-221.

[11] M. Sheik John – Theis " A study on genaralisation of closed sets and continuous maps in topological and bitopological spaces"

[12] P.Sundaram and S.John,On ω-closed sets in topology, Acta Ciencia Indica,4(2000)

[13] M.L. Thivagar, A note on quotient mappings, Bull. Malaysian math. Soc. 14 (1991),21-30.

[14] M.K.R.S. Veera Kumar, Between semi - closed sets and semi pre – closed sets, Rend. Istit. Mat. Univ. Trieste. XXXII (2000),25 – 41.