On new method of estimation of parameters of Size-biased Generalized Gamma Distribution and its Structural properties

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Abstract: In this paper, a new class of size-biased Generalized Gamma distribution is introduced. The estimates of parameters of size-biased Generalized Gamma distribution are obtained by using new method of moments. A new distribution which contains as a special case is introduced. The characterizing properties of the model are derived. Also, other important property including Shannon entropy which is the measure of the uncertainty in this class of the distribution are derived and studied.

Key words: Generalized Gamma distribution, Size-biased Generalized Gamma distribution, moment generating function, sample coefficient of variation, Shannon's entropy

I.   Introduction

The Generalized gamma distribution presents a flexible family in the varieties of shapes and hazard functions for modelling duration. It was introduced by Stacy [1]. Distributions that are used in duration analysis in economics include exponential [2, 3], lognormal [4], gamma [5], and Weibull [6]. The generalized gamma distribution, which encompasses exponential, gamma, and Weibull as subfamilies, and lognormal as a limiting distribution, has been used in economics by Jasggia [7]. In information theory, the important properties including Shannon entropy and Fisher's information matrix which are the measure of the uncertainty in this class of the distribution are derived and studied. The concept of Shannon entropy [8] is the central role of information theory, some-times referred as measure of uncertainty. The entropy of a random variable is defined in terms of its probability distribution and can be shown to be a good measure of randomness or uncertainty.

Although Prentice [9] have presented a procedure to obtain the three parameters of the generalized gamma distribution, his procedure still quit complicated. In this paper, we propose a simple procedure to obtained three estimators by using its characterization and moment estimation approach. Note that Hwang .T and Huang. P [10] have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized gamma distribution. Their characterization is used to derive the expectation and the variance of V_n^2 and then the new estimators for the three parameters of size-biased generalized gamma distribution are proposed.

The pdf of the generalized gamma distribution is given by:

$$f(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}} \ , for \ x>0 \ and \ \lambda,\beta,k>0 \qquad (1)$$

The mth non-central moments of GGD is given by

$$E(X^m)=\frac{\Gamma\left(k+\dfrac{m}{\beta}\right)}{\lambda^m \Gamma k} \qquad\qquad m=1,2,........$$

$$(2)$$

II.   Size biased Generalized Gamma Distribution

A size- biased Generalized Gamma distribution (SBGGMD) is obtained by applying the weights x^c, where c =1 to the Generalized Gamma distribution.

$$f_S(x;\lambda,\beta,k)=\int_0^{\infty}\frac{x\,f(x;\lambda,\beta,k)}{E(X)}\ dx$$

$$f_S(x;\lambda,\beta,k) = \int_0^\infty \frac{x \dfrac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1} e^{-(\lambda x)^\beta}}{\dfrac{\Gamma\left(k+\dfrac{1}{\beta}\right)}{\lambda\,\Gamma(k)}}\, dx$$

$$f_S(x;\lambda,\beta,k) = \int_0^\infty \frac{\lambda\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta}\, dx$$

Where $g_w(x;\lambda,\beta,k)$ represents a probability density function. This gives the size -biased generalized gamma distribution (SBGGMD) and its pdf is given by

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta}$$

(3)

For $\lambda > 0, k \geq 0, \beta > 0\ and\ \ k\beta > 1$

2.1   *Special cases*:

1.   When $\beta = k = 1,$ then Size biased Generalized Gamma distribution reduced to Size-biased exponential distribution and its probability distribution is given by

$$f_S(x;\lambda) = \lambda^2 x\, e^{-\lambda x}\quad ,\lambda > 0$$

(4)

2.   When $\beta = 1$ then Size biased Generalized Gamma distribution reduced to size-biased gamma distribution and its probability distribution is given by

$$f_S(x;\lambda,k+1) = \frac{\lambda^{k+1}}{\Gamma k+1}\, e^{-\lambda x}\, x^k\quad ,\lambda > 0, k \geq 0\ ,0 < x < \infty$$

(5)

3.   When $k = 0\, and\, \beta = 1$ then Size biased Generalized Gamma distribution reduced to exponential distribution and its probability distribution is given by

$$f_S(x;\lambda) = \lambda\ e^{-\lambda x}\quad ,\lambda > 0\ ,0 < x < \infty$$

(6)

Theorem 2.2. Let $X = (x_1, x_2, x_3...x_n)$ be an identical independently distributed sample from size-biased generalized gamma distribution

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta}$$

then, $E(X^m) = \dfrac{\beta\,\Gamma\left(k+\dfrac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\dfrac{1}{\beta}\right)}\qquad\qquad m = 1,2,........$

Proof: Using equation (2), the mth non central moment is given by,

$$E(X^m)\ =\ \int_0^\infty x^m \frac{\lambda\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)}(\lambda x)^{k\beta}\ e^{-(\lambda x)^\beta}\, dx$$

$$E(X^m)\ =\ \frac{\beta\,\lambda^{k\beta+1}}{\Gamma\left(k+\dfrac{1}{\beta}\right)}\int_0^\infty x^{(k\beta+r+1)-1}\ e^{-(\lambda x)^\beta}\, dx$$

On solving the above equation, we get

$$E(X^m) = \frac{\beta\,\Gamma\left(k+\frac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\frac{1}{\beta}\right)} \tag{7}$$

III. New Moment Estimator Of The Size-Biased Generalized Gamma Distribution

For deriving new moment estimators of three parameters of the size-biased generalized gamma distribution, we need the following theorem obtained by using the similar approach of Hwang .T and Huang .P (Theorems of 2006).

Theorem 3.1. Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random variables having a probability density function $f(x)$. Then the independence of the sample mean $\overline{X}_n$ and the sample coefficient of variation $V_n = \frac{S_n}{\overline{X}_n}$ is equivalent to that $f(x)$ is a size-biased generalized gamma density where S_n is the sample standard deviation.

The next theorem is easy to prove and need to derive the expectation and the variance of $V_n{}^2 = \left(\frac{S_n}{\overline{X}_n}\right)^2$, where $\overline{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 3.2: Let $n \geq 3$ and let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized gamma density

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\frac{1}{\beta}\right)}(\lambda x)^{k\beta}\,e^{-(\lambda x)^\beta}\ ,\ x > 0, \lambda > 0, \beta \geq 0, k > 0$$

Then $E(S_n{}^2) = \dfrac{\beta\left[\Gamma\left(k+\frac{1}{\beta}\right)\Gamma\left(k+\frac{3}{\beta}\right) - \beta\,\Gamma\left(k+\frac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{1}{\beta}\right)}$

Proof :

Here, $E(X^m) = \dfrac{\beta\,\Gamma\left(k+\frac{m+1}{\beta}\right)}{\lambda^m\,\Gamma\left(k+\frac{1}{\beta}\right)}\ ,\ m = 1,2,3,\ldots$

$$E(\overline{X}_n) = \frac{\beta\,\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda\,\Gamma\left(k+\frac{1}{\beta}\right)}$$

$$E\left(\overline{X}_n{}^2\right) = \frac{\beta\,\Gamma\left(k+\frac{3}{\beta}\right)\Gamma\left(K+\frac{1}{\beta}\right) + (n-1)\beta^2\Gamma^2\left(k+\frac{2}{\beta}\right)}{nk\,\Gamma^2\left(k+\frac{1}{\beta}\right)}$$

And $E(S_n{}^2) = n.V(\overline{X}_n)$

$$E(S_n{}^2) = \frac{\beta\left[\Gamma\left(k+\dfrac{1}{\beta}\right)\Gamma\left(k+\dfrac{3}{\beta}\right)-\beta\Gamma\left(k+\dfrac{2}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\dfrac{1}{\beta}\right)}$$

(8)

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Theorem 3.3: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized gamma density

$$f_S(x;\lambda,\beta,k) = \frac{\lambda\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)}(\lambda x)^{k\beta}\,e^{-(\lambda x)^{\beta}}\,,\ x>0, \lambda>0, \beta \geq 0, k>0 \qquad \text{Then}$$

$$E\left(\frac{S_n{}^2}{\overline{X}_n}\right) = \frac{n\left[\Gamma\left(k+\dfrac{3}{\beta}\right)\Gamma\left(k+\dfrac{1}{\beta}\right)-\beta\Gamma^2\left(k+\dfrac{2}{\beta}\right)\right]}{\Gamma\left(k+\dfrac{3}{\beta}\right)\Gamma\left(k+\dfrac{1}{\beta}\right)+(n-1)\beta\Gamma^2\left(k+\dfrac{2}{\beta}\right)}$$

Where $\overline{X}_n$ and $S_n{}^2$ are respectively their sample mean and sample variance.

Proof: By theorem 3.1, we have

$$E(S_n{}^2) = E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}.\overline{X}_n{}^2\right) = E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right)..E\left(\overline{X}_n{}^2\right)$$

And hence

$$E\left(\frac{S_n{}^2}{\overline{X}_n{}^2}\right) = \frac{E\left(S_n{}^2\right)}{E\left(\overline{X}_n{}^2\right)}$$

Applying theorem 3.2 to the above identity yields that

$$E\left(\frac{S_n{}^2}{\overline{X}_n}\right) = \frac{n\left[\Gamma\left(k+\dfrac{3}{\beta}\right)\Gamma\left(k+\dfrac{1}{\beta}\right)-\beta\Gamma^2\left(k+\dfrac{2}{\beta}\right)\right]}{\Gamma\left(k+\dfrac{3}{\beta}\right)\Gamma\left(k+\dfrac{1}{\beta}\right)+(n-1)\beta\Gamma^2\left(k+\dfrac{2}{\beta}\right)}$$

(9)

Thus 3.3 is established.

Note that $\qquad E\left(\dfrac{S_n{}^2}{\overline{X}_n}\right) \to \dfrac{\left[\Gamma\left(k+\dfrac{3}{\beta}\right)\Gamma\left(k+\dfrac{1}{\beta}\right)\right]}{\beta\Gamma^2\left(k+\dfrac{2}{\beta}\right)} - 1$ as $n \to \infty$ and that this limit is the square of

the population coefficient of variation. Thus, $\dfrac{S_n{}^2}{\overline{X}_n{}^2}$ is an asymptotically unbiased estimator of the square of the

population coefficient of variation.

Theorem 3.4: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized gamma density

$$f_S(x; \lambda, \beta, k) = \frac{\lambda \beta}{\Gamma\left(k + \frac{1}{\beta}\right)} (\lambda x)^{k\beta} \, e^{-(\lambda x)^{\beta}} \, , \, x > 0, \lambda > 0, \beta > 0, k \geq 0$$

Then
$$E\left(e^{tx^{\beta}}\right) = \frac{\lambda^{k\beta+1}}{\left(\lambda^{\beta} - t\right)^{k + \frac{1}{\beta}}}$$

Proof: Here $E\left(e^{tx^{\beta}}\right) = \int\limits_0^{\infty} e^{tx^{\beta}} f(x; \lambda, \beta, k) dx$

$$E\left(e^{tx^{\beta}}\right) = \int\limits_0^{\infty} e^{tx^{\beta}} \frac{\beta \lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} x^{k\beta} \, e^{-(\lambda x)^{\beta}} dx$$

$$E\left(e^{tx^{\beta}}\right) = \frac{\beta \lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} \int\limits_0^{\infty} x^{k\beta} \, e^{-x^{\beta}\left(\lambda^{\beta} - t\right)} dx$$

Put $x^{\beta} = t \Rightarrow x = t^{\frac{1}{\beta}} \qquad dx = \frac{dt}{\beta x^{\beta-1}}$

$$x \to 0, t \to 0 \quad and \quad x \to \infty, t \to \infty$$

$$E\left(e^{tx^{\beta}}\right) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} \int\limits_0^{\infty} t^k \, e^{-t\left(\lambda^{\beta} - t\right)} \frac{dt}{t^{1-\frac{1}{\beta}}}$$

$$E\left(e^{tx^{\beta}}\right) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} \int\limits_0^{\infty} t^{k + \frac{1}{\beta} - 1} \, e^{-t\left(\lambda^{\beta} - t\right)} dt$$

$$E\left(e^{tx^{\beta}}\right) = \frac{\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} \frac{\Gamma\left(k + \frac{1}{\beta}\right)}{\left(\lambda^{\beta} - t\right)^{k + \frac{1}{\beta}}}$$

$$E\left(e^{tx^{\beta}}\right) = \frac{\lambda^{k\beta+1}}{\left(\lambda^{\beta} - t\right)^{k + \frac{1}{\beta}}}$$

(10)

Substitute $\beta = 1$ in the above relation, we have

$$E\left(e^{tx}\right) = \frac{\lambda^{k+1}}{\left(\lambda - t\right)^{k+1}}$$

(11)

Shannon's entropy of size-biased Generalized Gamma Distribution

For deriving be entropy of the size-biased Generalized Gamma distribution, we need the following two definitions that more details of them can be found in [11].

Definition 3.2.1: The entropy of the discrete alphabet random variable f defined on the probability space is defined by:

$$H_P(f) = -\sum_{i=1}^{n} p(f = a) \log(p(f = a))$$

It is obvious that

$$H_P(f) \geq 0$$

Definition 3.2.1: The oblivious generalizations of the definition of entropy for a probability density function f defined on the real line as:

$$H(f) = -\int_0^\infty f(x) \log f(x) = E(-\log(x))$$

Provided the integral exists.

Theorem 3.5: Let $X_1, X_2, X_3 \ldots X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized gamma density

$$f_S(x; \lambda, \beta, k) = \frac{\lambda\beta}{\Gamma\left(k + \frac{1}{\beta}\right)}(\lambda x)^{k\beta} e^{-(\lambda x)^\beta}, \ x > 0, \lambda > 0, \beta > 0, k \geq 0$$

Then Shannon's entropy of Size-biased Generalized Gamma Distribution is:

$$H(f(x; \lambda, k, \beta)) = -\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)}\right) - k\left[\Psi\left(K + \frac{1}{\beta}\right) - \log\lambda^\beta\right] + \lambda^\beta\beta\frac{\Gamma\left(k + \frac{2}{\beta}\right)}{\lambda\Gamma\left(k + \frac{1}{\beta}\right)}$$

Proof: Shannon's entropy is defined as:

$$H[f(x; \alpha, \beta, k)] = E[-\log\{f(x; \alpha, \beta, k)\}]$$

$$H[f(x; \alpha, \beta, k)] = E\left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} x^{k\beta} e^{-(\lambda x)^\beta}\right\}\right]$$

$$H[f(x; \alpha, \beta, k)] = \left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)}\right\} - k\beta E(\log x) + \lambda^\beta\beta \ E(x)\right]$$

$$H[f(x; \alpha, \beta, k)] = \left[-\log\left\{\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)}\right\} - k\beta E(\log x) + \lambda^\beta\beta \ E(x)\right]$$

$$H(f(x; \lambda, k, \beta)) = -\log\left(\frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)}\right) - k\beta E(\log x) + \lambda^\beta\beta\frac{\Gamma\left(k + \frac{2}{\beta}\right)}{\lambda\Gamma\left(k + \frac{1}{\beta}\right)} \quad (12)$$

Now, $E(\log(x)) = \int_0^\infty \log x f(x; \alpha, \beta, k) dx$

$$E(\log(x)) = \int_0^\infty \log x \frac{\beta\lambda^{k\beta+1}}{\Gamma\left(k + \frac{1}{\beta}\right)} x^{k\beta} e^{-(\lambda x)^\beta} dx \quad (13)$$

Let $(\lambda x)^\beta = t \Rightarrow x^\beta = \dfrac{t}{\lambda^\beta} \Rightarrow x = \left(\dfrac{t}{\lambda^\beta}\right)^{\frac{1}{\beta}}$ $\qquad dx = \dfrac{dt}{\lambda \beta\, t^{1-\frac{1}{\beta}}}$

$$E(\log(x)) = \dfrac{\beta \lambda^{k\beta+1}}{\Gamma\left(k+\dfrac{1}{\beta}\right)} \int_0^\infty \log\left(\dfrac{t}{\lambda^\beta}\right)^{\frac{1}{\beta}} \left(\dfrac{t}{\lambda^\beta}\right)^{k\beta} e^{-t} \dfrac{dt}{\lambda \beta\, t^{1-\frac{1}{\beta}}}$$

$$E(\log(x)) = \dfrac{1}{\Gamma\left(k+\dfrac{1}{\beta}\right)} \int_0^\infty \log\left(\dfrac{t}{\lambda^\beta}\right)^{\frac{1}{\beta}} t^{k+\frac{1}{\beta}-1} e^{-t} dt$$

$$E(\log(x)) = \dfrac{1}{\beta\,\Gamma\left(k+\dfrac{1}{\beta}\right)} \int_0^\infty \left(\log t\, e^{-t}\, t^{k+\frac{1}{\beta}-1} dt\right) - \dfrac{\log \lambda^\beta}{\Gamma\left(k+\dfrac{1}{\beta}\right)} \int_0^\infty e^{-t} t^{k+\frac{1}{\beta}-1} dt$$

$$E(\log(x)) = \dfrac{\Gamma'\left(k+\dfrac{1}{\beta}\right)}{\beta\,\Gamma\left(k+\dfrac{1}{\beta}\right)} - \dfrac{\log \lambda^\beta\, \Gamma\left(k+\dfrac{1}{\beta}\right)}{\beta\,\Gamma\left(k+\dfrac{1}{\beta}\right)}$$

$$E(\log(x)) = \dfrac{\Psi\left(k+\dfrac{1}{\beta}\right) - \log \lambda^\beta}{\beta}$$

$$\text{(14)}$$

Substitute the value of equation (14) in equation (12), we have

$$H\big(f(x;\lambda,k,\beta)\big) = -\log\left(\dfrac{\beta \lambda^{k\beta+1}}{\Gamma\left(k+\dfrac{1}{\beta}\right)}\right) - k\left[\Psi\left(K+\dfrac{1}{\beta}\right) - \log \lambda^\beta\right] + \lambda^\beta \beta \dfrac{\Gamma\left(k+\dfrac{2}{\beta}\right)}{\lambda\,\Gamma\left(k+\dfrac{1}{\beta}\right)} \qquad \text{(15)}$$

The relation (15) is the Shannon's entropy of Size-biased Generalized Gamma Distribution.

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