On Bitopological $\gamma$-open sets

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Abstract: The purpose of this paper is to introduce and study the properties of $(1,2)$ $\gamma$-open set in bitopological space. In this paper we also study $(1,2)$ $\gamma$-locally closed set, $(1,2)$ $\gamma$-t-set, $(1,2)$ $\gamma$-$B$-set and relationship between $(1,2)$ $b$-open set and $(1,2)$ $\gamma$-open set.

Keywords:(1,2) $b$-open set, $(1,2)$ $B$-set, $(1,2)$ $t$-set (1,2) locally closed, (1,2) $\gamma$-open set.

I. Introduction

D. Andrijevic and M. Ganster [3] introduced a class of generalised open sets in a topological spaces, the so called $\gamma$-open set. The class of $\gamma$-open sets contains all semi-open sets, pre-open sets, and b-open sets. Further Tong [4] introduced the concept of t-set and B-set in topological spaces. In 1963 J.C. Kelly [5] introduced the concept of bitopological spaces. The purpose of this paper is to introduce the $\gamma$-open set in bitopological space, study the properties of this set and investigate the relationship between $(1,2)$ $\gamma$-open set, $(1,2)$ locally closed set, $(1,2)$ $\gamma$-locally closed set and $(1,2)$ b-open set. In this paper lastly we study comparison of $(1,2)$ $\gamma$-open set with $(1,2)^{\gamma}$ $\gamma$-open set.

II. Preliminaries

Throughout this paper by X we mean bitopological space $(X, \tau_1, \tau_2)$

Definition 2.1 A subset A of X is $\tau_1\tau_2$-(open) if $A \subseteq \tau_1\tau_2$ -open in X. The $\tau_1\tau_2$-closure of A is denoted by $\tau_1\tau_2$-cl(A) and $\tau_1\tau_2$-cl(A) = $\bigcap \{F : A \subseteq F \text{ and } F \text{ is } \tau_1\tau_2 \text{ -open}\}$

Remark 2.2 Notice that $\tau_1\tau_2$-open subsets of X need not necessarily form a topology.

Now, we recall some definitions and results which are used in this paper.

Definition 2.3 [7] A subset A of a bitopological space X is

(i) $(1,2)$ semi-open if $A \subseteq \tau_2$-cl($\tau_1$-int(A))
(ii) $(1,2)$ pre-open set if $A \subseteq \tau_1$-int($\tau_2$-cl(A))
(iii) $(1,2)$ $\alpha$-open set if $A \subseteq \tau_1$-int($\tau_2$-cl( $\tau_1$-int(A))$)
(iv) $(1,2)$ b-open set if $A \subseteq \tau_1$-int($\tau_2$-cl(A)) $\bigcup \tau_2$-cl($\tau_1$-int(A)).
(v) $(1,2)$ regular open set if $A = \tau_1$-int($\tau_2$-cl(A)).

The family of all $(1,2)$ open sets of X is called their respective closed sets. The family of all $(1,2)$ open sets of X is called their respective closed sets. The family of all $(1,2)$ open sets of X will be denoted by $(1,2)$ O(X), $(1,2)$ $\alpha$O(X), $(1,2)$ RO(X), $(1,2)$ aO(X), $(1,2)$ bO(X).

Proposition 2.4[7] In a bitopological space $(X, \tau_1, \tau_2)$ any open set in $(X, \tau_1)$ is $(1,2)$ b-open set and any open set in $(X, \tau_2)$ is $(2,1)$ b-open set.

Remark 2.5[7] In a bitopological space $(1,2)$ b-open set and $(1,2)$ locally closed are independent.

Proposition 2.6 [7]
(i) The union of any family of $(1,2)$ b-open sets is $(1,2)$ b-open.
(ii) The intersection of any $(1,2)$ open set and a $(1,2)$ b-open set is a $(1,2)$ b-open set.

III. $(1,2)$ $\square$ - open set

Definition 3.1 A subset A of $(X, \tau_1, \tau_2)$ is said to be $(1,2)$ $\gamma$-open set if for any non empty $(1,2)$ pre-open set $B$ in X such that $A \cap B \subseteq \tau_1$-int($\tau_2$-cl($A \cap B$))

Example 3.2 Let $X = \{a, b, c\}, \tau_1 = \{\{a\}, \{b\}, \{a, b\}, \phi, X\}$, \tau_2 = \{\{b\}, \phi, X\}$

$(1,2)$ $\gamma$-O(X) = $\{\{a\}, \{b\}, \{a, b\}, \{b,c\}, \phi, X\}$

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Remark 3.3 In a bitopological space $(X, \tau_1, \tau_2)$ a $\tau_1$ open set and a $(1, 2)$ $\gamma$-open set are independent as seen in the following examples.

Example 3.4 Let $X=\{a,b,c\}, \tau_1=\{\{a\}, \{a,c\}, \phi, X\}$

$$1, 2)PO(X)=\{\{a\}, \{a,c\}, \phi, X\}$$

$$(1, 2)\gamma-O(X) = \{\{a\}, \{b\}, \{a,b\}, \phi, X\}$$

Here, $\{a,c\}$ is $\tau_1$ open but not a $(1, 2)\gamma$-open set.

Example 3.5 Let $X=\{a,b,c\}, \tau_1=\{\{a\}, \{a,c\}, \phi, X\}$

$$1, 2)PO(X)=\{\{a\}, \{c\}, \{a,c\}, \phi, X\}$$

$$(1, 2)\gamma-O(X) = \{\{a\}, \{b\}, \{a,b\}, \phi, X\}$$

Here $\{b,c\}$ is a $(1, 2)\gamma$-open set but not a $\tau_1$-open set.

Definition 3.6 A subset $A$ of $(X, \tau_1, \tau_2)$ is called

(i) $(1, 2)$ locally closed if $A = U \cap V$, where $U \in \tau_1$ and $V$ is a $\tau_2$-closed set.

(ii) $(1, 2)$ locally $\gamma$-closed if $A = U \cap V$, where $U \in \tau_1$ and $V$ is a $\tau_2$-$\gamma$ closed set.

Remark 3.7 $(1, 2)\gamma$-open sets and $(1, 2)$ locally closed are independent as seen from the following example.

Example 3.8 Let $X=\{a,b,c,d\}, \tau_1=\{\{a\}, \{a,b\}, \{a,c\}, \{a,b,d\}, \{a, c,d\}, \phi, X\}$

$$1, 2)\gamma-O(X) = \{\{a\}, \{b\}, \{a,b\}, \phi, X\}$$

$$(1, 2)\gamma-LC(X) = \{\{a\}, \{b\}, \{a,b\}, \phi, X\}.$$ Here $\{a, c\}$ is $(1, 2)\gamma$-open and $\{a, c\}$ is $(1, 2)$ locally closed.

Remark 3.9 A space is called $(1, 2)$ extremally $\gamma$-disconnected if $\tau_2-\gamma$ closure of each $\tau_1-\gamma$ open set is $\tau_1-\gamma$ open, similarly $\tau_1-\gamma$ closure of each $\tau_2-\gamma$ open set is $\tau_2-\gamma$ open.

Example 3.10 Let $X=\{a,b,c\}, \tau_1=\{\{a\}, \{a,b\}, \{a,c\}, \phi, X\}$

$$\tau_1-\gamma-O(X) = \{\{a\}, \{b\}, \{a,b\}, \phi, X\}.$$ Here, every $\tau_2-\gamma$ closure of each $\tau_1-\gamma$ open set is $\tau_1-\gamma$ open set and also every $\tau_1-\gamma$ closure of each $\tau_2-\gamma$ open set is $\tau_2-\gamma$ open set.

Theorem 3.11 For a subset $A$ of an $(1, 2)$ $\gamma$-disconnected space $X$, if $A$ is $(1, 2)\gamma$-open set and $(1, 2)$ locally closed set, then $A$ is $\tau_1$-open.

Proof Let $A$ be $(1, 2)\gamma$-open set and $(1, 2)$ locally closed set. So

$$(A \cap B) \subseteq \tau_1 \text{ int}(\tau_2\text{Cl}(A \cap B)) \text{ and } A = U \cap \tau_2\text{Cl}(A).$$ Then

$$(A \cap B) \subseteq \tau_1 \text{ int}(\tau_2\text{Cl}(A \cap B)) \text{ and } A = U \cap \tau_2\text{Cl}(A).$$

Then

$$\tau_1 \text{ int}(\tau_2\text{Cl}(A \cap B)) \text{ and } A = U \cap \tau_2\text{Cl}(A).$$

Therefore $A$ is $\tau_1$-open.

Proposition 3.12 Let $H$ be a subset of $X$, $H$ is $(1, 2)$ locally $\gamma$-closed if there exist an $\tau_1$ open set $U \subseteq X$ such that $H = U \cap (1, 2)\gamma\text{ cl}(H)$.

Proof Since $H$ is $(1, 2)$ locally $\gamma$-closed $\Rightarrow H = U \cap F$, where $U$ is $\tau_1$ open and $F$ is $(1, 2)$ $\gamma$-closed. So $H \subseteq U, H \subseteq F$

$H \subseteq (1, 2)\gamma\text{ cl}(H) \subseteq (1, 2)\gamma\text{ cl}(F) = F$.

Hence $H \subseteq U \cap (1, 2)\gamma\text{ cl}(H) \subseteq U \cap (1, 2)\gamma\text{ cl}(F) = U \cap F = H$
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Hence \( H = U \cap (1,2) \gamma \text{-cl}(H) \).

Conversely since \((1,2) \gamma \text{-cl}(H)\) is \((1,2) \gamma\) closed and \( H = U \cap (1,2) \gamma \text{-cl}(H) \) then \( H \) is \((1,2) \gamma\) locally \( \gamma \) closed.

**Proposition 3.13** The union of any family of \((1,2) \gamma\) -open set is a \((1,2) \gamma\) -open set.

**Proof:** Let \( A \) and \( B \) be any two \((1,2) \gamma\) -open set. So there exist two non empty pre-open set \( C \) and \( D \), we have \((A \cap C) \subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(A \cap C)) \).

\[
(B \cap D) \subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(B \cap D))
\]

Now,
\[
(A \cup B) \cap (C \cap D) = (A \cap C) \cup (B \cap D)
\]
\[
\subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(A \cap B)) \cup \tau_1 \text{ int}(\tau_2 \text{cl}(B \cap D))
\]
\[
\subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(A \cup B)) \cap \tau_1 \text{ int}(\tau_2 \text{cl}(B \cup D))
\]
\[
\subseteq \tau_1 \text{ int}(\tau_2 \text{cl}((A \cup B) \cap (C \cup D))).
\]

Hence the union of any family of \((1,2) \gamma\) -open set is a \((1,2) \gamma\) -open set.

**Proposition 3.14** The intersection of any two \((1,2) \gamma\)-open set is a \((1,2) \gamma\) -open set.

**Proof** Let \( A \) and \( B \) be any two \((1,2) \gamma\) -open set.

So, for any \((1,2) \) pre open set \( C \)
\[
(A \cap C) \subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(A \cap B)) \text{ and }
\]
\[
(B \cap C) \subseteq \tau_1 \text{ int}(\tau_2 \text{cl}(B \cap C))
\]

Now,
\[
(A \cap B) = A \cap (B \cap C)
\]

By the definition of \((1,2) \gamma\) -open set \((A \cap (B \cap C))\) is a \((1,2) \) pre open set. Hence \( A \cap B \) is a \((1,2) \gamma\) -open set.

**Proposition 3.15** Let \( A \) be a subset of \((X, \tau_1, \tau_2)\) and if \( A \) is \((1,2) \) locally \( \gamma\) -closed then

(i) \((1,2) \gamma \text{cl}(A)-A \) is \((1,2) \gamma\) -closed.

(ii) \([A \cup (X- \gamma \text{cl}(A))]\) is \((1,2) \gamma\) -open set.

(iii) \( A \subseteq \gamma \text{int}(A \cup (X- \gamma \text{cl}(A)))\)

**Proof** (i) If \( A \) is an \((1,2) \) locally \( \gamma\) -closed, then there exist an \( \tau_1 \) open set U such that \( A = U \cap (1,2) \gamma \text{cl}(A) \)

Now, \( (1,2) \gamma \text{cl}(A)-A \)
\[
= (1,2) \gamma \text{cl}(A) - [U \cap (1,2) \gamma \text{cl}(A)]
\]
\[
= (1,2) \gamma \text{cl}(A) \cap [(X-U) \cup (X-(1,2) \gamma \text{cl}(A))]
\]
\[
= (1,2) \gamma \text{cl}(A) \cap (X-\text{U}), \text{which is (1,2) \gamma\text{closed. (by proposition 3.8) }
\]

(ii) Since \((1,2) \gamma \text{cl}(A)-A \) is \((1,2) \gamma\) -closed, then \([X-(1,2) \gamma \text{cl}(A)-A]\) is \((1,2) \gamma\) -open.

And
\[
[[X-(1,2)\text{cl}(A)-A]]
\]
\[
= (X-(1,2) \text{cl}(A)) \cup (X \cap A)
\]
\[
= A \cup [X-(1,2) \text{cl}(A)].
\]

Hence \([A \cup (X- \text{cl}(A))]\) is \((1,2) \gamma\) -open.

(iii) It is clear that
\[
A \subseteq [A \cup (X- \text{cl}(A))]
\]
\[
= (1,2) \gamma\text{ int }([A \cup (X- \gamma \text{cl}(A))]).\text{Hence the proof.}
\]

**Remark 3.16** \((1,2) \alpha\) -open set and \((1,2) \) locally \( \gamma\) closed are independent as seen from the following example.

**Example 3.17** Let \( X=\{a,b,c,d\} \), \( \tau_1=\{\{a\}, \{a,b\}, \{a,c,d\}, \varnothing, X\} \), \( \tau_2=\{\{a\}, \{a,d\}, X\} \).

\( (1,2) \alpha \) \( \text{O}(X)=\{\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, X\} \)

Locally \((1,2) \gamma \text{C}(X)=\{\{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{b,c\}, \{b,d\}, \{c,d\}, \{b,c,d\}, \{a,c,d\}, \varnothing, X\} \)

It is obvious that \( \{a,b,c\} \) is a \((1,2) \alpha\) -open set but not \((1,2) \gamma\) -closed. Also \( \{b,c,d\} \) is \((1,2) \gamma\) -closed but not \((1,2) \alpha\) -open set.
Proposition 3.18 A subset $A$ of $(X, \tau_1, \tau_2)$ is $\tau_1$-open iff it is $(1, 2)$ pre-open and $(1,2)$ B-set.

Proof If $A$ is $\tau_1$ open then $A$ is $(1,2)$ pre-open.

And $A \subseteq \tau_1 \cap (\tau_2 \cap (A))$

So $A = U \cap V$, where $U \in \tau_1$ and $V$ is a $(1,2)$ t-set. Hence $A$ is a $(1,2)$ pre-open and $(1,2)$ B-set. Conversely $A$ is a $(1,2)$ pre-open and $(1,2)$ B-set.

Therefore $\tau_1 \cap (\tau_2 \cap (A)) = U \cap V$, where $U \in \tau_1$ and $V$ is a $(1,2)$ t-set.

Hence $\tau_1 \cap (\tau_2 \cap (A))$

Also $\tau_1 \cap A \subseteq A$

Hence $A$ is $\tau_1$ open.

Definition 3.19 A subset $A$ of a space $(X, \tau_1, \tau_2)$ is said to be

(i) $(1, 2)$ $\gamma$-semi open if $A \subseteq \tau_2 \cap (\tau_1 \gamma \text{ int}(A))$

(ii) $(1, 2)$ $\gamma$-pre open if $A \subseteq \tau_1 \cap (\tau_2 \gamma \text{ cl}(A))$

Proposition 3.20 For subsets $A$ and $B$ of a space $(X, \tau_1, \tau_2)$ the following properties are hold.

(i) $A$ is $(1,2) \gamma$-t set iff it is $(1,2)$ $\gamma$-semi closed.

(ii) If $A$ and $B$ are $(1,2) \gamma$-t sets then $(A \cap B)$ is a $(1,2) \gamma$-t set.

(iii) If $A$ is $\tau_2 \gamma$ closed then it is a $(1,2) \gamma$-t set.

Proof (i) Let $A$ be $(1,2) \gamma$-t set.

So $\tau_1 \cap A = \tau_1 \cap (\tau_2 \gamma \text{ cl}(A))$

Therefore $\tau_1 \cap (\tau_2 \gamma \text{ cl}(A)) \subseteq \tau_1 \cap A \subseteq A$ and $A$ is $(1,2) \gamma$-semi closed.

Conversely, if $A$ is $(1,2) \gamma$-semi closed then

$\tau_1 \cap (\tau_2 \gamma \text{ cl}(A)) \subseteq A$

Thus $\tau_1 \cap (\tau_2 \gamma \text{ cl}(A)) \subseteq \tau_1 \cap A$

Also $A \subseteq \tau_2 \gamma \text{ cl}(A)$

$\Rightarrow \tau_1 \cap A \subseteq \tau_1 \cap (\tau_2 \gamma \text{ cl}(A))$

Hence $\tau_1 \cap A = \tau_1 \cap (\tau_2 \gamma \text{ cl}(A))$.

(ii) Let $A$ and $B$ be $(1,2) \gamma$-t sets. Then we have

$\tau_1 \cap (A \cap B) \subseteq \tau_1 \cap (\tau_2 \cap (A \cap B))$

$\subseteq \tau_1 \cap (\tau_2 \gamma \text{ cl}(A)) \cap \tau_1 \cap (\tau_2 \gamma \text{ cl}(B))$

$= \tau_1 \cap \tau_1 \cap A \cap B$

Then $\tau_1 \cap (A \cap B) = \tau_1 \cap (\tau_2 \gamma \text{ cl}(A \cap B))$

Hence $A \cap B$ is $(1,2) \gamma$-t set.

(iii) Let $A$ be a $\tau_2 \gamma$ closed.

$A = \tau_2 \gamma \text{ cl}(A)$

$\Rightarrow \tau_1 \cap A = \tau_1 \cap (\tau_2 \gamma \text{ cl}(A))$

Hence $A$ is a $(1,2) \gamma$-t set.

Remark 3.21 In a bitopological space, if $A$ is a $(1,2) \gamma$-t set then it is may not be $(1,2) \gamma$-closed as seen in the following example.

Example 3.22 Let $X = \{a, b, c\}$, $\tau_1 = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset, X\}$, $\tau_2 = \{\{b\}, \emptyset, X\}$

$(1, 2) \text{ PO}(X) = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset, X\}$, $(1, 2) \text{ O}(X) = \{\{a\}, \{b\}, \{a, b\}, \{a, b, c\}, \emptyset, X\}$

$(1, 2) \gamma$-cl$(X) = \{\{b\}, \{a\}, \{a, c\}, \{c\}, \{a\}, \emptyset, X\}$.

Here $\{b\}$ is $(1,2) \gamma$-t set but it is not $(1,2) \gamma$-closed.
Remark 3.23 In bitopological space every $(1, 2)$-$\gamma$-open set is $(1, 2)$ $b$-open set but converse is not true as seen in the following example.

**Example 3.24** Let $X = \{a, b, c, d\}$, $\tau_1 = \{\{a\}, \{a, c\}, \{a, b\}, \varphi, X\}$, $\tau_2 = \{\{a\}, \{d\}, \{a, d\}, \varphi, X\}$

\[
(1, 2) \gamma \text{O}(X) = \{\{a\}, \{a, b\}, \{a, d\}, \varphi, X\}
\]

\[
(1, 2) b \text{O}(X) = \{\{a\}, \{a, b\}, \{a, d\}, \varphi, X\}
\]

It is obvious that $(a, c)$ is $(1, 2)$ $b$-open set but not $(1, 2) \gamma$-open set.

**IV . $(1, 2)^*\gamma$ - Open Set**

Now we define $(1, 2)^*\gamma$-open set using the $\tau_{1,2}$ open set [8] and $\tau_{1,2}$ pre open set [8].

**Definition 4.1** Let $A$ be a subset of $X$. Then $A$ is called $\tau_{1,2}$-open set if $A = A_1 \cup B_1$.

Where $A_1 \in \tau_1$, $B_1 \in \tau_2$.

The complement of $\tau_{1,2}$-open set [8] is $\tau_{1,2}$-closed set. The family of all $\tau_{1,2}$-open set and $\tau_{1,2}$-closed set is denoted by $(1, 2)^-\text{O}(X)$, $(1, 2)^-\text{C}(X)$.

**Definition 4.2** Let $A$ be a subset of a bitopological space $X$. Then

1) $\tau_{1,2}$-closure of $A$ [8] denoted by $\tau_{1,2}$-cl$(A)$ is defined as the intersection of all $\tau_{1,2}$-closed sets containing $A$.

2) $\tau_{1,2}$-interior of $A$ [8] denoted by $\tau_{1,2}$-int $(A)$ is defined as the union of all $\tau_{1,2}$-open sets contained in $A$.

**Definition 4.3** A subset $A$ of a space $(X, \tau_1, \tau_2)$ is called $(1, 2)^*-\gamma$-open set if there exist a non empty $(1, 2)^*-\text{pre}$ open set $B$ such that $(A \cap B) \subseteq \tau_{1,2} - \text{int}(A \cap B))$.

**Example 4.4** Let $X = \{a, b, c\}$, $\tau_1 = \{\{a\}, \{a, b\}, \varphi, X\}$, $\tau_2 = \{\{c\}, \varphi, X\}$

\[
\tau_{1,2} \text{-open} = \{\{a\}, \{a, b\}, \{c\}, \varphi, X\}
\]

\[
(1, 2)^* \text{-open set} = \{\{a\}, \{a, b\}, \{c\}, \varphi, X\}
\]

**Remark 4.5** Every $(1, 2)^*\gamma$ open set is a $(1, 2)^*\gamma$-open set but converse may not be true as seen in the following example.

**Example 4.6** Let $X = \{a, b, c\}$, $\tau_1 = \{\{a\}, \{a, b\}, \{b\}, \varphi, X\}$, $\tau_2 = \{\{c\}, \{a, b\}, \varphi, X\}$

\[
(1, 2) \text{-pre-open} = \{\{a\}, \{a, b\}, \{b\}, \{c\}, \varphi, X\}
\]

\[
(1, 2)^*-\text{open set} = \{\{a\}, \{a, b\}, \{b\}, \varphi, X\}
\]

**V . Conclusion**

We know that in bitopological space collection of open set need not necessarily form a topology [2] and violate the topological properties. But from the present study we conclude that collection of $\gamma$-open set form a topology in bitopological space. Also in this paper we have proved $(1, 2)^*\gamma$-O$(X) \subseteq (1, 2)^*\gamma$-O$(X)$ and we will use the collection $(1, 2)^*\gamma$-open set for more results.

**References**