Closed sets in topological spaces

P.G.Palanimani¹, R.Parimelazhagan²
¹(Research Scholar,Karpagam University,India)
²(Department of Science and Humanities, Karpagam College of Engineering, Anna University, India)

Abstract: In this paper, the authors introduce and study the concept of a new class of closed sets called weakly generalized $\beta^*$ closed sets (briefly $\beta^*$ closed set). Also we investigate some of their properties

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I. Introduction

Levine.N[1970] introduced the concept of generalized closed (briey g-closed) sets in topological spaces. S.Arya and Nour[1990], Bhattacharya.P and Lahiri.B.K[1987],Levine.N[1963], Maki et.al[1982] introduced and investigated generalized semi-open sets, semi generalized open sets, generalized open sets, semi-open sets, pre-open sets and $\alpha$ - open sets, semi pre-open sets which are some of the weak forms of open sets and the complements of these sets are called the same types of closed sets.

Ever since general topologists extend the study of generalized closed sets on the basis of generalized semi-open sets, semi generalized open sets, generalized open sets, semi-open sets, pre-open sets and $\alpha$ - open sets, semi pre-open sets. Dontchev and Maki have introduced the concept of generalized closed sets. In 1997 Park et.al., introduced the notion of $\delta$ -semi open sets and investigated several properties of open sets. In 1986 Maki continued the work of Levine and Dunham on generalised closed sets and closure operations by introducing the notion of generalized $\Delta$ -set in topological spaces$(X, \tau)$.

Extension research of generalized closedness was done in recent years as the notion of generalized semi-open sets, semi generalized open sets, generalized open sets, semi-open sets, pre-open sets and $\alpha$ - open sets, semi pre-open sets were investigated. The aim of this paper is to continue the study of generalized closed sets in general and in particular, the notion of generalized $\beta^*$ closed sets and its various characterizations were studied.

II. Preliminaries

Before entering into our work we recall the following definitions in our sequel.

Definition 2.1 [1]: A subset A of a topological space $(X, \tau)$ is called a semi open set if $A \subseteq \text{cl} (\text{int} (A))$ and semi closed set if $\text{int} (\text{cl} (A)) \subseteq A$.

Definition 2.2 [2]: A subset A of a topological space $(X, \tau)$ is called a pre-open set if $A \subseteq \text{int} (\text{cl} (A))$ and pre-closed set if $\text{cl} (\text{int} (A)) \subseteq A$.

Definition 2.3 [3]: A subset A of a topological space $(X, \tau)$ is called an $\alpha$ -open set if $A \subseteq \text{int} (\text{cl} (A))$) and an $\alpha$ -closed set if $\text{cl} (\text{int} (A)) \subseteq A$.

Definition 2.4 [4]: A subset A of a topological space $(X, \tau)$ is called a semi-preopen set ($\beta$ -open set) if $A \subseteq \text{cl} (\text{int} (A))$ and semi-preclosed set if $\text{int} (\text{cl} (A)) \subseteq A$.

Definition 2.5 [5]: A subset A of a topological space $(X, \tau)$ is called a generalized closed set(briefly g- closed) if $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in X.

Definition 2.6 [6]: A subset A of a topological space $(X, \tau)$ is called a semi generalized closed set (simply sg-closed) if $\text{cl} (A) \subseteq U$, whenever $A \subseteq U$ and U is semi open in X.

Definition 2.7 [7]: A subset A of a topological space $(X, \tau)$ is called a generalized semi closed set (simply gs-closed) if $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is semi open in X.

Definition 2.8 [8]: A subset of A topological space $(X, \tau)$ is called a $\alpha$ -generalized closed set (briefly $\alpha$ g-closed) if $\alpha \text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is open in $(X, \tau)$.
**Definition 2.9** [9]: A subset $A$ of a topological space $(X, \tau)$ is called generalized $\alpha$-closed if $cl(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is $\alpha$ open in $(X, \tau)$.

**Definition 2.10** [10]: A subset $A$ of a topological space $(X, \tau)$ is called a generalized semi pre closed (briefly gsp -closed) if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.

**Definition 2.11** [11]: A subset $A$ of a topological space $(X, \tau)$ is called a weakly generalized closed set (briefly wgclosed) if $cl(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is open in $X$.

**Definition 2.12** [12]: A subset $A$ of a topological space $(X, \tau)$ is called semi weakly closed set(briefly swg-closed) if $cl(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is semi open in $X$.

**Definition 2.13** [13]: A subset $A$ of a topological space $(X, \tau)$ is called regular open set if $A = \text{int}(\text{cl}(A))$ and regular closed set $cl(\text{int}(A)) = A$.

### III. SOME BASIC PROPERTIES OF $\beta^*$-CLOSED SETS

In this section we introduce the concept of $\beta^*$ closed sets in topological space.

**Definition 3.1**: A subset $A$ of a topological space $(X, \tau)$ is called a weakly $\beta^*$ closed set if $cl(\text{int}(A)) \subseteq U$, whenever $A \subseteq U$ and $U$ is g open.

**Theorem 3.2**: If a subset $A$ of a topological space $X$ is g-closed then it is $\beta^*$ closed set in $X$.

**Proof**: Suppose $A$ is g-closed, let $U$ be an g open set containing $A$ in $X$, then $U \supseteq \text{cl}(A)$. Now $U \supseteq \text{cl}(A) \supseteq \text{cl}(\text{int}(A))$. Thus $A = \beta^*$ closed set.

**Remark**: The converse of the above theorem is true as seen from the following example.

**Example 3.3**: Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. In this topological space the subset $\{c\}$ is $\beta^*$ closed and g closed.

**Theorem 3.4**: A set $A$ is $\beta^*$ closed iff $cl(\text{int}(A))$ - $A$ containing no non empty closed set.

**Proof**: Suppose that $F$ is a non empty closed subset of $cl(\text{int}(A))$. Now $F \subseteq cl(\text{int}(A)) - A$, $F \subseteq cl(\text{int}(A)) \cap A^c$. Since $cl(\text{int}(A)) - A = cl(\text{int}(A)) \cap A^c$, then $F \subseteq cl(\text{int}(A))$. Here $F$ is open and $A$ is $\beta^*$ closed. We have $cl(\text{int}(A)) \subseteq F^c$. $F \subseteq cl(\text{int}(A)) \cap (cl(\text{int}(A)))^c = \phi$ implies $int(cl(A)) - A$ contains no nonempty closed set.

**Sufficiency**: Let $A \subseteq G$, $G$ is g-open. Suppose that $cl(\text{int}(A))$ is not contained in $G$, then $cl(\text{int}(A)) \cap G^c$ is a non empty closed set of $cl(\text{int}(A))$ - $A$ which is contradiction. Therefore $cl(\text{int}(A)) \subseteq G$ and hence $A$ is $\beta^*$ closed.

**Theorem 3.5**: Suppose that $B \subseteq A \subseteq X$, $B$ is $\beta^*$-closed set relative to $A$ and that $A$ is both g-open and $\beta^*$ closed subset of $X$, then $B$ is $\beta^*$-closed set relative to $X$.

**Proof**: Let $B \subseteq G$ and $G$ be a $g$-open set in $X$. But given that $B \subseteq A \subseteq X$, therefore $B \subseteq A$ and $B \subseteq G$. This implies $B \subseteq A \cap G$. Since $B$ is $\beta^*$-closed relative to $A$, $cl(B) \subseteq A \cap G$, i.e. $A \cap G$ implies $A \cap cl(B) \subseteq G$. Thus $A \cap cl(B) \subseteq G$. Therefore $cl(B) \subseteq G$. Also $B \subseteq A$ implies $cl(int(B)) \subseteq cl(int(A))$. Thus $cl(B) \subseteq cl(int(A))$. Therefore $cl(B) \subseteq G$, since $cl(B)$ is not contained in $cl(int(B))$. Thus $B$ is $\beta^*$ closed set relative to $X$.

**Corollary 3.6**: Let set $A$ be $\beta^*$ closed and suppose $F$ is closed then $A \cap F$ is $\beta^*$ closed.

**Proof**: To show $A \cap F$ is $\beta^*$ closed, we have to show that $cl(int(A)) \subseteq U$, whenever $A \cap F \subseteq U$, g-open, $A \cap F$ is closed in $A$ and so $\beta^*$ closed in $A$. Therefore $A \cap F$ is $\beta^*$ closed in $X$, since $A \cap F \subseteq A \subseteq X$.

**Theorem 3.7**: If $A$ is $\beta^*$ closed set and $A \subseteq B \subseteq cl(int(A))$ then $B$ is $\beta^*$ closed.

**Proof**: Given that $B \subseteq cl(int(A))$, then $cl(B) \subseteq cl(int(A))$, then $cl(B) - B \subseteq cl(int(A)) - A$. Therefore
A ⊆ B, A is $\beta^*$ closed. Then by theorem $\text{cl} (\text{int}(A)) - A$ containing no non empty set, $\text{cl} (\text{int}(B)) - B$ containing no non empty set, B is $\beta^*$ closed.

We have the following implications for properties of subsets:

IV. $\beta^*$ CLOSED SETS

**Theorem 4.1:** If a subset A of a topological space X is $\beta^*$ closed then it is gsp-closed but not conversely.

**Proof:** Suppose A is $\beta^*$ closed set in X, Let U be g-open containing A then $\text{cl} (\text{int}(A)) \subseteq U$, $\text{int} (\text{cl}(\text{int}(A))) \subseteq U$, U is open which implies $A \cup \text{int} (\text{cl}(\text{int}(A))) \subseteq A \cup U$, that is $\text{spcl}(A) \subseteq U$, then A is gsp-closed in X.

The converse of the above theorem need not be true as seen from the following example.

**Example 4.2:** Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. In this topological space the subset $\{a\}$ is gsp-closed but not $\beta^*$ closed set in X.

**Remark 4.3:** If A and B are $\beta^*$ closed set. Then $A \cap B$ will be a $\beta^*$ closed set. Consider $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ In this topological space the subset $A = \{b, c\}, B = \{c\}$ are $\beta^*$ closed sets in X. $A \cap B = \{c\}$ is also a $\beta^*$ closed set.

**Theorem 4.4:** Every closed set in a topological space X is $\beta^*$ closed.

**Proof:** Suppose A is closed set in X, Let U be g-open set containing A in X such that $A \subseteq U$ and $A \subseteq \text{cl}(\text{int}(A))$ $\subseteq U$, clearly $\text{gcl}(A) \subseteq \text{cl}(A) \subseteq U$ that implies $A \subseteq \text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \subseteq U$, U is g-open, thus $A \subseteq \text{gcl}(A) \subseteq U$, A is $\beta^*$ closed.

**Remark:** The converse of the above theorem is true as seen from the following example.

**Example 4.5:** Let $X = \{a, b, c\}$ with the topology $\tau = \{X, \phi, \{a\}, \{a, b\}\}$. In this topological space the subset $\{c\}$ is $\beta^*$ closed set in X and also closed in $(X, \tau)$.

**Theorem 4.6:** Every g-closed set in a topological space is a $\beta^*$ closed.

**Proof:** Suppose A is g-closed set in X, Let U be g-open set containing A in X such that $A \subseteq U$ and $A \subseteq \text{cl}(\text{int}(A))$ $\subseteq U$, clearly $\text{gcl}(A) \subseteq \text{cl}(A) \subseteq U$ that implies $A \subseteq \text{cl}(\text{int}(A)) \subseteq \text{cl}(A) \subseteq U$, U is g-open, thus $A \subseteq \text{gcl}(A) \subseteq U$, A is $\beta^*$ closed.

**Remark:** The converse of the above theorem is true as seen from the following example.
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**Example 4.7:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,c\}\}$. In this topological space the subset $\{b\}$ is $\beta^+$ closed set in $X$ and also $g$-closed in $(X,\tau)$.

**Theorem 4.8:** Every $\beta^+$ closed set in a topological space is rgw-closed.

**Proof:** The set $\text{cl}(\text{int}(A)) \subseteq U$ whenever $A \subseteq U \cup \{U\}$ is semi open, Every open set is regular open, so by definition every $\beta^+$ closed set in a topological space is rgw-closed.

**Remark 4.9:** The converse of the above theorem need not be true from the following example.

**Example 4.10:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,b\}\}$. In this topological space the subset $\{a,b\}$ is rgw-closed set in $X$ but not $\beta^+$ closed set in $(X,\tau)$.

**Remark 4.11:** Every $\beta^+$ closed set is sg-closed set.

**Example 4.12:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\}\}$. In this topological space the subset $\{a\}$ is sg-closed but not $\beta^+$ closed set in $X$.

**Remark 4.13:** Every $\beta^+$ closed set is $\alpha g$-closed set.

**Example 4.14:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{b\}\}$. In this topological space the subset $\{b\}$ is g closed but not $\beta^+$ closed set in $X$.

**Remark 4.15:** Every $\beta^+$ closed set is g $\alpha$ closed set.

**Example 4.16:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,c\}\}$. In this topological space the subset $\{a,c\}$ is $g \alpha$ closed but not $\beta^+$ closed set in $X$.

**Remark 4.17:** Every $\beta^+$ closed set is gsp-closed set.

**Example 4.18:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,b\}\}$. In this topological space the subset $\{a,b\}$ is gsp-closed set but not $\beta^+$ closed set in $X$.

**Remark 4.19:** Every $\beta^+$ closed set is wg-closed set.

**Example 4.20:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{b\}\}$. In this topological space the subset $\{b\}$ is wg-closed set but not $\beta^+$ closed set in $X$.

**Remark 4.21:** Every $\beta^+$ closed set is swg-closed set.

**Example 4.22:** Let $X=\{a,b,c\}$ with the topology $\tau=\{X,\phi,\{a\},\{a,b\}\}$. In this topological space the subset $\{a,b\}$ is swg-closed set but $\beta^+$ closed set in $X$.

**REFERENCES**


