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On a basic integral formula involving Generalized Mellin - Barnes Type of contour integrals

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Abstract: *The aim of the present paper is to study some new unified integral formulas associated with the $\overline{H}$ which was introduced by Inayat Hussain. In this paper we evaluated finite double integral involving $\overline{H}$ function with general arguments and new finite integral of Generalized Mellin- Barnes Type of contour integrals. . These formulas are unified in nature and act as the key formulas from which we can obtain as their special cases.*

Keywords $\overline{H}$ *-function, generalized Wright hyper geometric function, Fox's H- function.*
AMS Classification: 26A33, 33C60.

I. Introduction

In 1987, Inayat- Hussain[1] was introduced generalization from of fox's H-function , which is popularly known as $\overline{H}$. $\overline{H}$ function is defined and represented in the following manner.

$$\bar{H}^{m,n}_{p,q}[z] = \bar{H}^{m,n}_{p,q}\left[z\Big|^{(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}}_{(b_j,\beta(j);B_j)_{1,m},(b_j,\beta_j)m+1,q}\right] = \frac{1}{2\pi\iota}\int_L \bar{\phi}(\xi) z^{\xi} d\xi.$$

$$(z \neq 0)$$

(1.1)
Where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j\xi)\prod_{j=1}^{n}\{\Gamma(1 - a_j + \alpha_j\xi)\}^{A_j}}{\prod_{j=m+1}^{q}\{\Gamma(1 - b_j + \beta_j\xi)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j\xi)}$$

(1.2)

It may be noted that the $\bar{\phi}(\xi)$ contains fractional powers of some of the gamma function and m, n,p,q are integers such that $1 \le m \le q$, $1 \le n \le p$ $(\alpha_j)_{1,p}$, $(\beta_j)1, q$ are positive real numbers and $(A_j)_{1,n}$, $(B_j)_{m+1,q}$ may take non -integer values , which we assume to be positive for standardization purpose $(a_j)_{1,p}$, $(b_j)1, q$ are complex number. The nature of contour L, sufficient conditions of convergence of defining integral (1.1) and other details about $\overline{H}$. The $\overline{H}$ -function can be seen in the paper [6].

The behavior of $\overline{H}$ for small values of $|z|$ follows easily from a result given by Rathie [11]:

$$\bar{H}^{m,n}_{p,q}[z] = O(|z|^{\alpha});$$

Where

$$\alpha = min_{1\le j\le m} Re(\frac{b_j}{\alpha_j}), |z| \to 0$$

(1.3)

$$\mu_1 = \sum_{j=1}^{m} |B_j| + \sum_{j=m+1}^{q} |b_j B_j| - \sum_{j=1}^{n} |a_j A_j| - \sum_{j=n+1}^{q} |A_j| > 0, 0 < |z| < \infty \quad (1.4)$$

The following function which follows as special cases of the $\overline{H}$ - function will be required and defined as follows;

$${}_p\psi_q\left[{}^{(a_j,\alpha_j,;A_j)_{(1,p)}}_{(b_j,\beta_j,;B_j)_{(1,q)}}; -z\right] = H_{p,q}^{m,n}\left[z\Big|{}^{(1-a_j,\alpha_j;A_j)_{1,p}}_{(0,1)(1-b_j,\beta(j);B_j)_{1,q}}\right] \quad (1.5)$$

We shall require the following formulas for the evaluation of our main integrals.

(i)Finite Integral {Erdelyi [1953] }

$$\int_0^{\frac{\pi}{2}} e^{i(\alpha+\beta)\theta} (sin\theta)^{\alpha-1} (cos\theta)^{\beta-1} d\theta = e^{\pi-\frac{i\alpha}{2}} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (1.6)$$

Valid for Re (α) > 0, Re (β) > 0.

(ii)Infinite Integrals {Erdelyi [1953]}

$$\int_0^{\infty} x^{\gamma-1/2}[(x+a)(x+b)]^{-\gamma} dx = \sqrt{\pi}(\sqrt{a}+\sqrt{b})^{1-2\gamma} \frac{\Gamma(\gamma-1/2)}{\Gamma(\gamma)} .$$

Valid for Re (γ) >½. (1.7)

(iii) Rainville [1971]

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1} dx = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho+\sigma)} . \quad (1.8)$$

Valid for Re (ρ) > 0, Re (σ) > 0.

II. Main results:

First Integral:

$$\int_0^{\infty} \int_0^{\infty} e^{\iota(\alpha+\beta)\theta} (\sin\theta)^{\alpha-1} (\cos\theta)^{\beta-1} x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} \bar{H}_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{-\lambda}|{}^{\dots;\dots;}_{\dots;\dots;}]dxd\theta$$

$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} \bar{H}_{p+2,q+2}^{m,n+2}\left[z\Big|{}^{(1-\rho,\delta),\dots,\dots(\nu,\lambda)}_{(\nu-1/2,\lambda),\dots,\dots(1-\alpha-\beta,\delta)}\right] \quad (2.1)$$

The above result will be converge under the following conditions:

γ > 0, δ> 0,Re(β) > 0, $|org z|$< 1/2Bπ

Where B is given by

$$\sum_{j=1}^{n}\alpha_j - \sum_{j-n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}B_{ji};$$

Second Integral:

$$\int_0^\infty \int_0^\infty e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu}\bar{H}_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{\lambda}|_{\ldots,\ldots}^{\ldots,\ldots}]dxd\theta$$

$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu}\bar{H}_{p+2,q+2}^{m,n+2}[z|_{\ldots,\ldots,(1-\nu,\lambda)(1-\sigma-\beta,\delta)}^{(3/2,-\nu,\lambda),(1-\beta,\delta),\ldots,\ldots}].$$

(2.2)

The above result will be converge under the following conditions:

$\gamma > 0$, $\delta > 0$, $Re(\beta) > 0$, $|org\, z| < 1/2B\pi$

Where B is given by

$$\sum_{j=1}^{n}\alpha_j - \sum_{j-n+1}^{p_i}\alpha_{ji} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}B_{ji};$$

III. Proof:

To establish first integral we express $\overline{H}$ occurring on the Left -hand-side of equation (2.1) in terms of Mellin - Barnes type of contour integral given by equation (1.1) we obtain (2.1).

To establish (2.2) replace $\overline{H}$ -function by its equivalent contour integral as given in equation, change the order of integration which is justifiable due to given condition we get second integral.

3.1 Special case: If we put Aj = Bj = 1, $\overline{H}$ function reduces to Fox's H-function, then the equation (2.1) and (2.2) takes the following form.

$$:\int_0^\infty \int_0^\infty e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu}H_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{-\lambda}|_{\ldots,\ldots}^{\ldots,\ldots}]dxd\theta$$

$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu}H_{p+2,q+2}^{m,n+2}[z|_{(\nu-1/2,\lambda),\ldots,\ldots(1-\alpha-\beta,\delta)}^{(1-\rho,\delta),\ldots,\ldots(\nu,\lambda)}]$$

(4.1.1)

$$\int_0^\infty \int_0^\infty e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} H_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{\lambda}|_{\ldots,\ldots}^{\ldots,\ldots}]dxd\theta$$
$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} H_{p+2,q+2}^{m,n+2}[z|_{\ldots,\ldots,(1-\nu,\lambda)(1-\sigma-\beta,\delta)}^{(3/2,-\nu,\lambda),(1-\beta,\delta),\ldots,\ldots}] \quad (4.1.2)$$

The Conditions of validity of (4.1.1) and (4.1.2) easily follow from those given in (2.1) and (2.2).

3.2: If we putAj = Bj = 1, αj = βj = 1 in (1.1) , $\overline{H}$ function reduces to Meijer's G -function [7] i. e.

$$\bar{H}_{p,q}^{m,n}[z|_{(b_j,1,1)_{1,m},(b_j,1)m+1,q}^{(a_j,1,1)_{1,n},(a_j,1)_{1,p}}] = G_{p,q}^{m,n}[z|_{(b_j,1)1,q}^{(a_j,1)_{1,p}}].$$

$$:\int_0^{\infty}\int_0^{\infty} e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} G_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{-\lambda}|_{\ldots,\ldots}^{\ldots,\ldots}]dxd\theta$$
$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} G_{p+2,q+2}^{m,n+2}[z|_{(\nu-1/2,\lambda),\ldots,\ldots(1-\alpha-\beta,\delta)}^{(1-\rho,\delta),\ldots,\ldots(\nu,\lambda)}] \quad (4.2.1)$$

$$\int_0^{\infty}\int_0^{\infty} e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} G_{p,q}^{m,n}[ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\}^{\lambda}|_{\ldots,\ldots}^{\ldots,\ldots}]dxd\theta$$
$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} G_{p+2,q+2}^{m,n+2}[z|_{\ldots,\ldots,(1-\nu,\lambda)(1-\sigma-\beta,\delta)}^{(3/2,-\nu,\lambda),(1-\beta,\delta),\ldots,\ldots}] \quad (4.2.2)$$

The Conditions of validity of (4.2.1) and (4.2.2) easily follow from those given in (2.1) and (2.2).

3.3: IF we put n = p,m = 1, q = q+1, b1 = 0, β1 = 1, aj = 1−aj , bj = 1−bj , in (1.1) then the $\overline{H}$ function reduces to generalized Wright hypergeometric function [16] i.e.

$$\bar{H}_{p,q}^{m,n}[z|_{(0,1)(1-b_j,\beta(j);B_j)_{1,q}}^{(1-a_j,\alpha_j;A_j)_{1,p}}] = {}_p\psi_q[_{(b_j,\beta_j,;B_j)_{(1,q)}}^{(a_j,\alpha_j,;A_j)_{(1,p)}}; -z]$$

Using same assumptions in the equations in the equations (2.1) , (2.2) then they takes the following form .

$$:\int_0^{\infty}\int_0^{\infty} e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} p\bar{\psi}_q\left[{}^{(a_j,\alpha_j,;A_j)_{(1,p)}}_{(b_j,\beta_j,;B_j)_{(1,q)}}; -Ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\left\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\right\}^{-\lambda}\right]dxd\theta$$

(3.3.1)

$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} \quad {}_{p+1}\bar{\psi}_{q+1}\left[{}^{(1-\rho,\delta),\ldots,\ldots(\nu,\lambda)}_{(\nu-1/2,\lambda),\ldots,\ldots(1-\alpha-\beta,\delta)}; -z\right]$$

$$:\int_0^\infty \int_0^\infty e^{\iota(\alpha+\beta)\theta}(\sin\theta)^{\alpha-1}(\cos\theta)^{\beta-1}x^{\nu-\frac{1}{2}}$$

$$[(x+a)(x+b)]^{-\nu} p\bar{\psi}_q\left[{}^{(a_j,\alpha_j,;A_j)_{(1,p)}}_{(b_j,\beta_j,;B_j)_{(1,q)}}; -ze^{\iota\delta\vartheta}(\cos\theta)^{\delta}\left\{\frac{x(\sqrt{a}+\sqrt{b})^2}{(x+a)(x+b)}\right\}^{\lambda}\right]dxd\theta$$

(3.3.2)

$$= \sqrt{\pi}e^{\iota\pi\alpha/2}\Gamma(\alpha)(\sqrt{a}+\sqrt{b})^{1-2\nu} \ {}_{p+1}\bar{\psi}_{q+1}\left[{}^{3/2,-\nu,\lambda),(1-\beta,\delta),\ldots,\ldots}_{\ldots,\ldots,(1-\nu,\lambda)(1-\sigma-\beta,\delta)}; -z\right]$$

The Conditions of validity of (4.3.1) and (4.3.2) easily follows from those given in (2.1) and (2.2).

IV. Conclusion:

In this paper, we have presented two integral formulas. The first results have been developed associated with H-function with general arguments. The results obtained in the present paper are useful in applications. These results will be useful to analysis the various problems in different field.

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