Semi-Compatible Maps On Intuitionistic Fuzzy Metric Space

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Abstract: In this paper, we prove common fixed point theorem for semi-compatible mappings on intuitionistic fuzzy metric space with different some conditions of Park and Kim ([10], 2008). This research extended and generalized the results of Singh and Chauhan ([14], 2000). The concept of fuzzy set was developed extensively by many authors and used in various fields. Several authors have defined fuzzy metric space Kramosil and Michalek(((5],1975) etc.) with various methods to use this concept in analysis. Jungck (([3],1986), ([4],1988)) researched the more generalized concept compatibility than commutativity and weak commutativity in metric space and proved common fixed point theorems, and Singh and Chauhan ([14],2000) introduced the concept of compatibility in fuzzy metric space and studied common fixed point theorems for self maps in intuitionistic fuzzy metric space.

I. Introduction:

In this paper, we prove common fixed point theorem for semi-compatible mappings on intuitionistic fuzzy metric space with different some conditions of Park and Kim ([10], 2008). This research extended and generalized the results of Singh and Chauhan ([14], 2000).

We give some definitions and properties of intuitionistic fuzzy metric space. Throughout this paper, N will denote the set of all positive integers.

Let us recall Schweizer and Sklar (see ([13], 1960)) that a continuous t-norm is a binary operation*: [0, 1] x [0, 1] → [0, 1] which satisfies the following conditions:
(a) * is commutative and associative;
(b) * is continuous;
(c) a * 1 = a for all a ∈ [0, 1];
(d) a * b ≤ c * d whenever a ≤ c and b ≤ d (a, b, c, d ∈ [0, 1]).

Similarly, a continuous t-conorm is a binary operation: [0, 1] x [0, 1] → [0, 1] which satisfies the following conditions:
(a) ◊ is commutative and associative;
(b) ◊ is continuous;
(c) a ◊ 0 = a for all a ∈ [0, 1];
(d) a ◊ b ≥ c ◊ d whenever a ≤ c and b ≤ d (a, b, c, d ∈ [0, 1]).

Also, let us recall (see [6] that the following conditions are satisfied:
(a) For any any r₁, r₂ ∈ (0, 1) with r₁ > r₂ there exist r₅, r₆ ∈ (0, 1) such that r₁ * r₅ ≥ r₂ and r₆ ◊ r₅ ≤ r₁;
(b) For any r₅ ∈ (0, 1), there exist r₆, r₇ ∈ (0, 1) such that r₆ * r₇ ≥ r₅ and r₇ ◊ r₅ ≤ r₅.

1.1 Definition:- (Park and Kwun ([7], 2006)). The 5-tuple (X, M, N, *, ◊) is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norms, ◊ is a continuous t-conorm and M, N are fuzzy sets on X such that
(a) M(x, y, t) > 0,
(b) M(x, y, t) = 1 ↔ x = y,
(c) M(x, y, t) = M(y, x, t),
(d) M(x, y, t) * M(y, z, s) ≤ M(x, z, t + s),
(e) M(x, y, ) : (0, ∞) → [0, 1] is continuous,
(f) N(x, y, t) > 0,
(g) N(x, y, t) = 0 ↔ x = y,
(h) N(x, y, t) = N(y, x, t),

www.iosrjournals.org 59 | Page
(i) \( N(x, y, t) \cap N(y, z, s) \geq N(x, z, t + s) \).
(ii) \( N(x, y, \cdot) : (0, \infty) \rightarrow (0, 1) \) is continuous.

Note that \( (M, N) \) is called an intuitionistic fuzzy metric on \( X \). The functions \( M(x, y, t) \) and \( N(x, y, t) \) denote the degree of nearness and the degree of non-nearness between \( x \) and \( y \) with respect to \( t \), respectively.

### 1.2 Definition
(Park and Kwun ([12], 2005)). Let \( X \) be an intuitionistic fuzzy metric space. Then (a) A sequence \( \{x_n\} \subset X \) is convergent to \( x \) in \( X \) if and only if for each \( \varepsilon > 0 \), \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x, t) > 1 - \varepsilon \), \( N(x_n, x, t) < \varepsilon \) for all \( n \geq n_0 \).

(b) A sequence \( \{x_n\} \subset X \) is called Cauchy sequence if and only if for each \( \varepsilon > 0 \), \( t > 0 \), there exists \( n_0 \in \mathbb{N} \) such that \( M(x_n, x_m, t) > 1 - \varepsilon \), \( N(x_n, x_m, t) < \varepsilon \) for all \( m, n \geq n_0 \).

(c) \( X \) is complete if every Cauchy sequence in \( X \) is convergent.

### 1.3 Definition
(Park and Kim ([10], 2008)). Let \( A, B \) be mappings from intuitionistic fuzzy metric space \( X \) into itself.

(a) \( (A, B) \) are said to be compatible if and only if
\[
\lim_{n \to \infty} M(ABx_n, BAx_n, t) = 1, \quad \lim_{n \to \infty} N(ABx_n, BAx_n, t) = 0, \quad \text{for all} \ t > 0, \quad \text{whenever} \ \{x_n\} \subset X
\]
such that \( \lim_{n \to \infty} n \to \infty A_x = \lim_{n \to \infty} B x = x \) for some \( x \in X \).

(b) \( (A, B) \) are said to be semi-compatible if and only if
\[
\lim_{n \to \infty} M(ABx_n, Bx, t) = 1, \quad \lim_{n \to \infty} N(ABx_n, Bx, t) = 0, \quad \text{for all} \ t > 0, \quad \text{whenever} \ \{x_n\} \subset X
\]
such that \( \lim_{n \to \infty} n \to \infty A_x = \lim_{n \to \infty} B x = x \) for some \( x \in X \).

### 1.4 Lemma
(Park([10],2008)). Let \( A, B \) to be self mappings on intuitionistic fuzzy metric space \( X \). If \( B \) is continuous, then \( (A,B) \) is semi-compatible if and only if \( (A,B) \) is compatible.

### II. Main Result

#### 2.1 Theorem
Let \( P, Q, S \) and \( T \) be self maps of complete intuitionistic fuzzy metric space \( X \) with \( t - \text{norm}^* \) and \( t - \text{conorms} \) (defined by \( a + b = \min \{a, b\} \) and \( a \cdot b = \max \{a, b\}, a, b \in [0, 1] \), satisfying

(a) \( (P, S) \) and \( (Q, T) \) are semi-compatible pairs of maps,

(b) \( S \) and \( T \) are continuous,

(c) \( P^p(x) \subset T^q(x), \quad Q^p(x) \subset S^q(X), \)

(d) \( M(P^p x, Q^q y, kt) \geq \min \{M(S^q x, T^p y, t), M(P^p x, S^q y, t), M(Q^q x, T^p y, t), M(P^p x, T^q y, \alpha t), M(Q^q y, S^p x, (2 - \alpha t)t)\} \),

\[
N(P^p y, Q^q x, k t) \leq \max \{N(S^q y, T^p x, t), N(P^p x, S^q y, t), N(Q^q y, T^p x, t), N(P^p x, T^q y, \alpha t), N(Q^q y, S^p x, (2 - \alpha t)t)\}.
\]

(e) \( \lim_{t \to \infty} t \to \infty M(x, y, t) = 1, \)
\( \lim_{t \to \infty} M(x, y, t) = 0 \)
for all \( x, y \in X, \alpha \in (0, 2), t > 0 \) and \( p, q, s, t \in \mathbb{N} \).

Then \( P, Q, S \) and \( T \) have a unique common fixed point in \( X \).

#### Proof
Let \( x_0 \) be an arbitrary point in \( X \), we can inductively construct a sequence \( \{y_n\} \subset X \) such that
\[
y_{2n+1} = T^x_{x_{2n+1}} = P^p y_{2n+2}, \quad y_{2n} = S^q y_{2n+1} \quad \text{for} \quad n = 1, 2, 3, ...
\]
First, we prove that \( \{y_n\} \) is a Cauchy sequence, from (d) with \( \alpha = 1 \), we have.

\[
M(y_{2n+1}, y_{2n+2}, K t) = M(P^p_{2n+2}, Q^q_{2n+2}, K t) \geq \min \{M(S^q_{2n+2}, T^p_{2n+2}, 1), M(P^p_{2n+2}, S^q_{2n+2}, 1), M(Q^q_{2n+2}, T^p_{2n+2}, 1), M(P^p_{2n+2}, T^q_{2n+2}, 1), M(Q^q_{2n+2}, S^p_{2n+2}, 0)\}
\]
\[
\geq \min \{M(y_{2n+1}, y_{2n+2}, 1), M(y_{2n+1}, y_{2n+2}, 1), M(y_{2n+2}, y_{2n+1}, 1), M(y_{2n+1}, y_{2n+2}, 1), M(y_{2n+2}, y_{2n+1}, 1)\}
\]
\[
\geq \min \{M(y_{2n+1}, y_{2n+2}, 1), M(y_{2n+2}, y_{2n+1}, 1), 1\}
\]
\[
N(y_{2n+1}, y_{2n+2}, K t) = (P^p_{2n+2}, Q^q_{2n+2}, K t).
\]
Semi-Compatible Maps On Intuitionistic Fuzzy Metric Space

\[ \leq \max \{ N(S^x_{2n}, T^x_{2n+1}, t), N(P^p_{2n}, S^x_{2n}, t), N(Q^q_{2n+1}, T^x_{2n+1}, t), N(P^p_{2n}, T^x_{2n+1}, t), N(Q^q_{2n+1}, S^x_{2n}, t) \} \]

\[ \leq \max \{ N(y_{2n}, y_{2n+1}, t), N(y_{2n+1}, y_{2n}, t), N(y_{2n+2}, y_{2n+1}, t), N(y_{2n+1}, y_{2n+1}, t), N(y_{2n+2}, y_{2n+1}, t) \} \]

\[ \leq \max \{ N(y_{2n}, y_{2n+1}, t), N(y_{2n+2}, y_{2n+1}, t), 0 \} \]

which implies

\[ M(y_{2n}, y_{2n+1}, L_k t) \geq M(y_{2n-1}, y_{2n}, t), \]

\[ N(y_{2n}, y_{2n+1}, L_k t) \leq N(y_{2n-1}, y_{2n}, t), \]

\[ \text{Generally,} \quad M(y_{n}, y_{n+1}, L_k t) \geq M(y_{n-1}, y_{n}, t), \]

\[ N(y_{n}, y_{n+1}, L_k t) \leq N(y_{n-1}, y_{n}, t). \]

Therefore,

\[ M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, \frac{L}{k}) \]

\[ \geq ... \]

\[ \geq M(y_0, y_1, \frac{L}{k^n}) \]

Taking limit \( n \to \infty \) then it tends to \( \to 1 \) as

\[ N(y_n, y_{n+1}, t) \leq N(y_{n-1}, y_n, \frac{L}{k}) \]

\[ \leq ... \]

\[ \leq N(y_0, y_1, \frac{L}{k^n}) \to 0 \text{ as } n \to \infty \]

Hence for \( t > 0 \) and \( \varepsilon \in (0, 1) \), we can choose \( n_0 \in \mathbb{N} \) such that

\[ M(y_n, y_{n+1}, t) > 1 - \varepsilon, \quad N(y_n, y_{n+1}, t) < \varepsilon \]

for all \( n \geq n_0 \).

Suppose that for \( m \),

\[ M(y_n, y_{n+m}, t) > 1 - \varepsilon, \quad N(y_n, y_{n+m}, t) < \varepsilon \]

for all \( n \geq n_0 \) and for every \( m \in \mathbb{N} \).

Then

\[ M(y_n, y_{n+m+1}, t) \geq \min \{ M(y_n, y_{n+m}, \frac{L}{2}), M(y_{n+m}, y_{n+m+1}, \frac{L}{2}) \} \]

\[ > 1 - \varepsilon, \]

\[ N(y_n, y_{n+m+1}, t) \leq \max \{ N(y_n, y_{n+m}, \frac{L}{2}), N(y_{n+m}, y_{n+m+1}, \frac{L}{2}) \} \]

\[ < \varepsilon. \]

Therefore \( \{y_n\} \subset X \) is a cauchy sequence.

Second, we prove that \( P^p, Q^q, S^x, \) and \( T^t \) have a unique common fixed point.

Since \( \{y_n\} \) converges to some point \( x \) from completeness of \( X \),

\[ P^p x_{2n} \to x, \quad S^x x_{2n} \to x, \quad Q^q x_{2n+1} \to x \] and \( T^t x_{2n+1} \to x \)

Since \( S \) is continuous, hence

\[ S^x (P^p x_{2n}) \to S^x(x) \]

Thus for \( t > 0 \) and \( \varepsilon \in (0, 1) \), there exists an \( n_0 \in \mathbb{N} \) such that

\[ M(S^x (P^p x_{2n}), S^x(x), \frac{L}{2}) > 1 - \varepsilon, \]

\[ N(S^x (P^p x_{2n}), S^x(x), \frac{L}{2}) < \varepsilon \]

for all \( n \geq n_0 \). Also since \( (P, S) \) and \( (Q, T) \) are semi – compatible pairs, by Lemma 1.4, \( (P, S) \) and \( (Q, T) \) are compatible pairs.

Therefore \( (P^p, S^x) \) and \( (Q^q, T^t) \) are compatible pairs for all \( P, q, s, t \in \mathbb{N} \). From (a), we have

\[ \lim n \to \infty \quad M(P^p (S^x x_{2n}), S^x(P^p x_{2n}), \frac{L}{2}) = 1 \]

www.iosrjournals.org 61 | Page
\[ \lim n \to \infty \ N(P^0(S^t \ x_{2n}), S^i(P^0_{2na}) \frac{L}{2}) = 0 \]

Hence,
\[ M(S^t(P^0_{2na}), S^i(t), t) \geq \min \{ M(P^0(S^t \ x_{2n}), S^i(P^0_{2na}), \frac{L}{2}), M(S^t(P^0_{2na}), S^i(t), \frac{L}{2}) \} \]
\[ > 1 - \varepsilon , \]
\[ N(S^t(P^0_{2na}), S^i(t), t) \leq \max \{ N(P^0(S^t \ x_{2n}), S^i(P^0_{2na}), \frac{L}{2}), N(S^t(P^0_{2na}), S^i(t), \frac{L}{2}) \} \]
\[ < \varepsilon \]

for all \( n \geq n_0 \).

Therefore \( \lim n \to \infty \ (P^0 S^i \ x_{2n}) = S^i \ x \).

Also since \( \lim n \to \infty \ Q^i(t_{2n-1}) = x \) and \( T \) is continuous,
\[ \lim n \to \infty \ T^i(t_{2n-1}) = T^i \ x. \]

Thus for \( t > 0 \) and \( \varepsilon \in (0, 1) \), there exists an \( n_0 \in \mathbb{N} \) such that
\[ M(T^i(Q^i(t_{2n-1}), T^i(x), t/2) > 1 - \varepsilon, N(T^i(Q^i(t_{2n-1}), T^i(x), t/2) < \varepsilon \]

for all \( n \geq n_0 \).

From (a), we have
\[ \lim n \to \infty \ M(Q^i(T^i \ x_{2n-1}), T^i(Q^i \ x_{2n-1}), t/2) = 1 \]
\[ \lim n \to \infty \ N(Q^i(T^i \ x_{2n-1}), T^i(Q^i \ x_{2n-1}), t/2) = 0 \]

Hence
\[ M(Q^i(T^i \ x_{2n-1}), T^i(x), t) \geq \max \{ M(Q^i(T^i \ x_{2n-1}), T^i(Q^i \ x_{2n-1}), t/2), M(T^i(Q^i \ x_{2n-1}), T^i(x), t) \} \]
\[ \geq 1 - \varepsilon \]
\[ N(Q^i(T^i \ x_{2n-1}), T^i(x), t) \leq \max \{ N(Q^i(T^i \ x_{2n-1}), T^i(Q^i \ x_{2n-1}), t/2), N(T^i(Q^i \ x_{2n-1}), T^i(x), t) \} \]
\[ \leq \varepsilon \]

for all \( n \geq n_0 \).

Therefore \( \lim n \to \infty \ Q^i(T^i \ x_{2n-1}) = T^i \ x. \)

Using (d) with \( \alpha = 1 \), we have
\[ M(P^0(S^i \ x_{2na}), Q^i(T^i \ x_{2na}), K) \geq \min \{ M(S^i(S^i \ x_{2na}), T^i(T^i \ x_{2na}), t), M(P^0(S^i \ x_{2na}), S^i(S^i \ x_{2na}), t), M(Q^i(T^i \ x_{2na}), T^i(T^i \ x_{2na}), t), M(P^0(S^i \ x_{2na}), T^i(T^i \ x_{2na}), t), M(Q^i(T^i \ x_{2na}), S^i(S^i \ x_{2na}), t) \} \]
\[ N(P^0(S^i \ x_{2na}), Q^i(T^i \ x_{2na}), K) \leq \max \{ N(S^i(S^i \ x_{2na}), T^i(T^i \ x_{2na}), t), N(P^0(S^i \ x_{2na}), S^i(S^i \ x_{2na}), t), N(Q^i(T^i \ x_{2na}), T^i(T^i \ x_{2na}), t), N(P^0(S^i \ x_{2na}), T^i(T^i \ x_{2na}), t), N(Q^i(T^i \ x_{2na}), S^i(S^i \ x_{2na}), t) \} \]

Taking limit as \( n \to \infty \) and using above results,
\[ M(S^i \ x_{2na}, T^i \ x_{2na}, K) \geq \min \{ M(S^i \ x_{2na}, T^i \ x_{2na}, t), M(S^i \ x_{2na}, S^i \ x_{2na}, t), M(T^i \ x_{2na}, T^i \ x_{2na}, t), M(S^i \ x_{2na}, T^i \ x_{2na}, t), M(T^i \ x_{2na}, S^i \ x_{2na}, t) \} \]
\[ \geq M(S^i \ x_{2na}, T^i \ x_{2na}, t) \]
\[ N(S^i \ x_{2na}, T^i \ x_{2na}, K) \leq \max \{ N(S^i \ x_{2na}, T^i \ x_{2na}, t), N(S^i \ x_{2na}, S^i \ x_{2na}, t), N(T^i \ x_{2na}, T^i \ x_{2na}, t), N(S^i \ x_{2na}, T^i \ x_{2na}, t), N(T^i \ x_{2na}, S^i \ x_{2na}, t) \} \]
\[ \leq N(S^i \ x_{2na}, T^i \ x_{2na}, t) \]

which implies
\[ S^i \ x_{2na} = T^i \ x_{2na} \]

Now from (d) with \( \alpha = 1 \),
\[ M(P^0 \ x_{2na}, Q^i(T^i \ x_{2na}), K) \geq \min \{ M(S^i \ x_{2na}, T^i \ x_{2na}, t), M(P^0 \ x_{2na}, S^i \ x_{2na}, t), M(Q^i(T^i \ x_{2na}), T^i(T^i \ x_{2na}), t), M(P^0 \ x_{2na}, T^i(T^i \ x_{2na}), t), M(Q^i(T^i \ x_{2na}), S^i(S^i \ x_{2na}), t) \} \]
\[ N(P^0 \ x_{2na}, Q^i(T^i \ x_{2na}), K) \leq \max \{ N(S^i \ x_{2na}, T^i(T^i \ x_{2na}), t), N(P^0 \ x_{2na}, S^i \ x_{2na}, t), N(Q^i(T^i \ x_{2na}), T^i(T^i \ x_{2na}), t), N(P^0 \ x_{2na}, T^i(T^i \ x_{2na}), t), N(Q^i(T^i \ x_{2na}), S^i(S^i \ x_{2na}), t) \} \]

Taking the limit as \( n \to \infty \) and using above results
\[ M(P^0 \ x_{2na}, T^i \ x_{2na}, K) \geq \min \{ M(T^i \ x_{2na}, T^i \ x_{2na}, t), M(P^0 \ x_{2na}, T^i \ x_{2na}, t), M(T^i \ x_{2na}, T^i \ x_{2na}, t), M(P^0 \ x_{2na}, T^i \ x_{2na}, t), M(T^i \ x_{2na}, T^i \ x_{2na}, t), M(P^0 \ x_{2na}, T^i \ x_{2na}, t) \} \]
\[ \geq M(P^0 \ x_{2na}, T^i \ x_{2na}, t) \]
\[ N(P^0 \ x_{2na}, T^i \ x_{2na}, K) \leq \max \{ N(T^i \ x_{2na}, T^i \ x_{2na}, t), N(P^0 \ x_{2na}, T^i \ x_{2na}, t), N(T^i \ x_{2na}, T^i \ x_{2na}, t), N(P^0 \ x_{2na}, T^i \ x_{2na}, t), N(T^i \ x_{2na}, T^i \ x_{2na}, t) \} \]
\[ M(p^\alpha_x, Q^\alpha_x, K t) \geq \max \{ M(p^\alpha_x, Q^\alpha_x, t), M(p^\alpha_x, S^\alpha_x, t), M(Q^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, T^\alpha_x, t), M(Q^\alpha_x, S^\alpha_x, t) \} \]

Hence \( p^\alpha_x = Q^\alpha_x = S^\alpha_x = T^\alpha_x \).

Furthermore using (d) with \( \alpha = 1 \), we have

\[ N(p^\alpha_x, Q^\alpha_x, K t) \geq \max \{ N(p^\alpha_x, Q^\alpha_x, t), N(p^\alpha_x, S^\alpha_x, t), N(Q^\alpha_x, T^\alpha_x, t), N(p^\alpha_x, T^\alpha_x, t), N(Q^\alpha_x, S^\alpha_x, t) \} \]

Taking limit as \( n \to \infty \) we have

\[ M(x, Q^\alpha_x, K t) \geq \max \{ M(x, Q^\alpha_x, t), M(x, Q^\alpha_x, t), M(Q^\alpha_x, Q^\alpha_x, t), M(x, Q^\alpha_x, t), M(Q^\alpha_x, x, t) \} \]

\[ N(x, Q^\alpha_x, K t) \leq \min \{ N(x, Q^\alpha_x, t), N(x, Q^\alpha_x, t), N(Q^\alpha_x, Q^\alpha_x, t), N(x, Q^\alpha_x, t), N(Q^\alpha_x, x, t) \} \]

Which implies \( x = Q^\alpha_x \).

Therefore \( x = Q^\alpha_x = T^\alpha_x \).

That is, \( x \) is a common fixed point of \( p^\alpha_x \), \( Q^\alpha_x \) and \( T^\alpha_x \). Let \( z \) be another common fixed point of maps. Then from (d) with \( \alpha = 1 \)

\[ M(p^\alpha_x, Q^\alpha_x, K t) \geq \max \{ M(p^\alpha_x, S^\alpha_x, t), M(p^\alpha_x, S^\alpha_x, t), M(Q^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, T^\alpha_x, t), M(Q^\alpha_x, S^\alpha_x, t) \} \]

\[ N(p^\alpha_x, Q^\alpha_x, K t) \leq \min \{ N(x, Q^\alpha_x, t), N(x, Q^\alpha_x, t), N(Q^\alpha_x, Q^\alpha_x, t), N(x, Q^\alpha_x, t), N(Q^\alpha_x, x, t) \} \]

Which implies \( x = z \).

Hence \( x \) is a unique common fixed point of maps.

Third, we prove that this point \( x \) is a common fixed point of \( P, Q \) and \( T \).

Since \( p_x = P(p^\alpha_x) = P^\alpha(P_x) \) and \( P_x = P(S^\alpha_x) = S^\alpha(P_x) \)

from (a), hence \( p_x \) is a common fixed point of \( p^\alpha_x \) and \( S^\alpha_x \). Also since \( Q_x = Q(Q^\alpha_x) = Q^\alpha(Q_x) \) and \( Q_x = Q(T^\alpha_x) = T^\alpha(Q_x) \) from (a), hence \( Q_x \) is a common fixed point of \( Q^\alpha_x \) and \( T^\alpha_x \). Now letting \( x = P_x \) and \( y = Q_x \) and \( \alpha = 1 \) in (d), we have

\[ M(P_x, Q_x, K t) = M(p^\alpha_x, Q^\alpha_x, K t) \geq \max \{ M(S^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, S^\alpha_x, t), M(Q^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, T^\alpha_x, t), M(Q^\alpha_x, S^\alpha_x, t) \} \]

\[ N(P_x, Q_x, K t) = N(p^\alpha_x, Q^\alpha_x, K t) \leq \min \{ N(S^\alpha_x, T^\alpha_x, t), N(p^\alpha_x, S^\alpha_x, t), N(Q^\alpha_x, T^\alpha_x, t), N(p^\alpha_x, T^\alpha_x, t), N(Q^\alpha_x, S^\alpha_x, t) \} \]

Therefore \( P_x = Q_x \).

Also from (d) with \( \alpha = 1 \), we have

\[ M(S^\alpha_x, T^\alpha_x, K t) = M(S^\alpha_x, T^\alpha_x, K t) \geq \max \{ M(S^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, S^\alpha_x, t), M(Q^\alpha_x, T^\alpha_x, t), M(p^\alpha_x, T^\alpha_x, t), M(Q^\alpha_x, S^\alpha_x, t) \} \]

\[ N(S^\alpha_x, T^\alpha_x, K t) = N(S^\alpha_x, T^\alpha_x, K t) \leq \min \{ N(S^\alpha_x, T^\alpha_x, t), N(p^\alpha_x, S^\alpha_x, t), N(Q^\alpha_x, T^\alpha_x, t), N(p^\alpha_x, T^\alpha_x, t), N(Q^\alpha_x, S^\alpha_x, t) \} \]
Therefore, $Sx = Tx$. Since $x$ is a unique common fixed point of $P^0$, $Q^0$, $S^0$, $T^0$. Hence $Px = Qx$ is a common fixed points of $P^0$, $S^0$ and $Sx = Tx$ is a common fixed points of $Q^0$, $T^0$. Hence $x = Px = Qx = Sx = Tx$. That is, $x$ is common fixed point of $P$, $Q$, $S$ and $T$.

References