IOSR Journal of Mathematics (IOSR-JM)
ISSN: 2278-5728. Volume 4, Issue 5 (Jan. - Feb. 2013), PP 43-53
www.iosrjournals.org

Prime Radicals in Ternary Semigroups

Y. Sarala [1], A.Anjaneyulu [2] D.Madhusudhana Rao [3]
[1]*Dept. of Mathematics, Nagarjuna University, Guntur, A.P. India.*
[2,3]*Dept. of Mathematics, V S R & N V R College, Tenali, A.P. India.*

Abstract: *In this paper the terms completely prime ideal, prime ideal, m-system. globally idempotent , semi simple elements of a ternary semigroup are Introduced. It is proved that an ideal A of a ternary semigroup T is completely prime if and only if T\A is either sub semigroup of T or empty. It is proved that if T is a globally idempotent ternary semigroup then every maximal ideal of T is a prime ideal of T. In this paper the terms completely semiprime ideal, semiprime ideal, n-system, d-system and i-system are introduced. It is proved that the non-empty intersection of any family of a completely prime ideal and prime ideal of a ternary semigroup T is a completely semiprime ideal of T. It is also proved that an ideal A of a ternary semigroup T is completely semiprime if and only if T\A is a d-system of T or empty. It is proved that if N is an n-system in a ternary semigroup T and* $a \in N$*, then there exist an m-system M in T such that* $a \in M$ *and* $M \subseteq N$*. The terms radical, complete radical of a ternary semigroup are introduced. It is proved that if A and B are any two ideals of a ternary semigroup T, then i)* $A \subseteq B \Rightarrow \sqrt{A} \subseteq \sqrt{B}$ *ii)* $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ *iii)* $\sqrt{\sqrt{A}} = \sqrt{A}$. *It is also proved that if A is an ideal of ternary semigroup T then* $\sqrt{A}$ = $\{x \in T : M(x) \cap A \neq \varnothing\}$.

Mathematics Subject Classification *(2010) : 20N12 , 20N05, 20N10, 20N12*
Keywords: *Ideal completely prime ideal, prime ideal, m-system, n-system, d-system, i-system, globally idempotent, semisimple elements, completely semiprime ideal, semi prime ideal, irreducible ideal, strongly irreducible ideal, radical, complete radical of ternary semigroup.*

I. Introduction

The theory of ternary algebraic system was introduced by Lehmer [13] in 1932, but earlier such structures were studied by Kasner [10] who gave the idea of n-ary algebras . Ternary semigroups are universal algebras with one associative ternary operation. Anjaneyulu.A [1],[2] initiated the study of ideals in semigroups. S.Kar and B.K.Maity [9] initiated the study of some ideals of ternary semigroups. Sioson. F. M [18] studied about Ideal theory in ternary semigroups. Iampan . A.[7] gave the idea of Lateral ideals of ternary semigroups.

II. Preliminaries

DEFINITION 2.1 : Let T be a non-empty set. Then T is said to be a ***ternary semigroup*** if there exist a mapping from T×T×T to T which maps $(x_1, x_2, x_3) \rightarrow [x_1x_2x_3]$ satisfying the condition : $[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$ $\forall$ $x_i \in$ T, $1 \leq i \leq 5$.

DEFINITION 2.2 : A ternary semigroup T is said to be ***commutative*** provided for all $a,b,c \in$ T, we have $abc = bca = cab = bac = cba = acb$.

DEFINITION 2.3 : An element *a* of ternary semigroup T is said to be ***left identity*** of T provided $aat = t$ for all $t \in$ T.

NOTE 2.4 : Left identity element *a* of a ternary semigroup T is also called as ***left unital element.***

DEFINITION 2.5 : An element *a* of a ternary semigroup T is said to be a ***lateral identity*** of T provided $ata = t$ for all $t \in$ T.

NOTE 2.6 : Lateral identity element *a* of a ternary semigroup T is also called as ***lateral unital element.***

DEFINITION 2.7 : An element *a* of a ternary semigroup T is said to be a ***right identity*** of T provided $taa = t$ $\forall$ $t \in$ T.

NOTE 2.8 : Right identity element *a* of a ternary semigroup T is also called as ***right unital element.***

DEFINITION 2.9 : An element *a* of a ternary semigroup T is said to be a ***two sided identity*** of T provided $aat = taa = t$ $\forall$ $t \in$ T.

NOTE 2.10 : Two-sided identity element of a ternary semigroup T is also called as ***bi-unital element.***

DEFINITION 2.11 : An element a of a ternary semigroup T is said to be an ***identity*** provided $aat = taa = ata = t \; \forall \; t \in T$.
NOTE 2.12: An identity element of a ternary semigroup T is also called as ***unital element.***
NOTE 2.13 : An element a of a ternary semigroup T is said to be an ***identity*** of T then a is left identity , lateral identity and right identity of T.
NOTATION 2.14 : Let T be a ternary semigroup. If T has an identity, let T^1 = T and if T does not have an identity , let T^1 be the ternary semigroup T with an identity adjoined usually denoted by the symbol 1.
DEFINITION 2.15 : Let T be ternary semigroup. A non empty subset S of T is said to be a ***ternary subsemigroup*** of T if $abc \in S$ for all $a,b,c \in S$.
NOTE 2.16 : A non empty subset S of a ternary semigroup T is a ternary subsemigroup if and only if SSS $\subseteq$ S.
DEFINITION 2.17 : Let T be a nonempty set. A nonempty finite sequence $a_1, a_2, \ldots a_{2n-1}$ usually written by juxtaposition $a_1a_2 \ldots\ldots a_{2n-1}$ of elements of T is called ***word*** over the alphabet T. The set T of all words with the operation of juxtaposition $(a_1a_2 \ldots\ldots a_{2p-1})(b_1b_2 \ldots\ldots b_{2q-1})(c_1c_2 \ldots\ldots c_{2r-1}) = a_1a_2 \ldots\ldots a_{2p-1} b_1b_2 \ldots\ldots b_{2q-1} \, c_1c_2 \ldots\ldots c_{2r-1}$ is a ternary semigroup called the ***free ternary semigroup*** over the alphabet T.
DEFINITION 2.18 : A nonempty subset A of a ternary semigroup T is said to be ***left ideal*** of T if $b, c \in$ T, $a \in$ A implies $bca \in$ A.
NOTE 2.19 : A nonempty subset A of a ternary semigroup T is said to be a left ideal of T if and only if TTA $\subseteq$ A.
DEFINITION 2.20 : A nonempty subset of a ternary semigroup T is said to be a ***lateral ideal*** of T if $b, c \in$ T , $a \in$ A implies $bac \in$ A.
NOTE 2.21 : A nonempty subset of A of a ternary semigroup T is a lateral ideal of T if and only if TAT $\subseteq$ A.
DEFINITION 2.22 : A nonempty subset A of a ternary semigroup T is a ***right ideal*** of T if $b, c \in$ T , $a \in$ A implies $abc \in$ A
NOTE 2.23 : A nonempty subset A of a ternary semigroup T is a right ideal of T if and only if ATT $\subseteq$ A.
DEFINITION 2.24 : A non-empty subset A of a ternary semigroup T is said to be ***ternary ideal*** or simply an ***ideal*** of T if $b, c \in$ T , $a \in$ A implies bca $\in$ A, $bac \in$ A, $abc \in$ A.
NOTE 2.25 : A nonempty subset A of a ternary semigroup T is an ideal of T if and only if it is left ideal, lateral ideal and right ideal of T.
DEFINITION 2.26 : An ideal A of a ternary semigroup T is said to be a ***proper ideal*** of T if A is different from T.
DEFINITION 2.27 : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by J (a) (or) < a >.
DEFINITION 2.28 : An ideal A of a ternary semigroup T is said to be a ***maximal left ideal*** provided A is a proper left ideal of T and is not properly contained in any proper left ideal of T.
DEFINITION 2.29 : An ideal A of a ternary semigroup T is said to be a ***maximal lateral ideal*** provided A is a proper lateral ideal of T and is not properly contained in any proper lateral ideal of T.
DEFINITION 2.30 : An ideal A of a ternary semigroup T is said to be a ***maximal right ideal*** provided A is a proper right ideal of T and is not properly contained in any proper right ideal of T.
DEFINITION 2.31 : An ideal A of a ternary semigroup T is said to be a ***maximal two sided ideal*** provided A is a proper two sided ideal of T and is not properly contained in any proper two sided ideal of T.
DEFINITION 2.32 : An ideal A of a ternary semigroup T is said to be a ***maximal ideal*** provided A is a proper ideal of T and is not properly contained in any proper ideal of T.
DEFINITION 2.33 : A left ideal A of a ternary semigroup T is said to be the ***principal left ideal generated by a*** if A is a left ideal generated by $\{a\}$ for some a $\in$ T. It is denoted by L (a) or $< a >_l$.

THEOREM 2.34 : If T is a ternary semigroup and $a \in$ T then L (a) = $a \cup$ TTa.

NOTE 2.35 : if T is ternary semigroup and $a \in$ T then L(a) = T^1T^1a .
DEFINITION 2.36 : A lateral ideal A of a ternary semigroup T is said to be the ***principal lateral ideal generated by a*** if A is a lateral ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by M (a) (or) $< a >_m$.

THEOREM 2.37 : If T is a ternary semigroup and $a \in$ T then M (a) = $a \cup$ TaT $\cup$ TTaTT.
DEFINITION 2.38 : A right ideal A of a ternary semigroup T is said to be a ***principal right ideal generated by a*** if A is a right ideal generated by $\{a\}$ for some $a \in$ T. It is denoted by R (a) (or) $< a >_r$.

THEOREM 2.39 : If T is a ternary semigroup and a $\in$T then R (a) = a$\bigcup$ aTT.

NOTE 2.40 : If T is a ternary semigroup and $a \in$T then R (a) = a T^1T^1
DEFINITION 2.41 : A two sided ideal A of a ternary semigroup T is said to be the ***principal two sided ideal*** provided A is a two sided ideal generated by $\{a\}$ for some
$a \in$T. It is denoted by T (a) (or) < a >$_t$.
THEOREM 2.42 : If T is a ternary semigroup and *a* $\in$T then T (*a*) = *a*$\bigcup$ TT*a*$\bigcup$ *a*TT$\bigcup$ TT*a*TT.
DEFINITION 2.43 : An ideal A of a ternary semigroup T is said to be ***a principal ideal*** provided A is an ideal generated by $\{a\}$ for some $a \in$T. It is denoted by J (a) (or) < a >.
THEOREM 2.44 : If T is a ternary semigroup and a $\in$T then
J (*a*) = *a*$\bigcup$ *a*TT$\bigcup$ TT*a*$\bigcup$ T*a*T$\bigcup$ TT*a*TT.
NOTE 2.45 : If T is a ternary semigroup and a $\in$T then
J (a) = $a\bigcup a$TT$\bigcup$ TT$a\bigcup$ TaT$\bigcup$ TTaTT = $T^1\, T^1\, a T^1\, T^1$.

III. Completely Prime Ideals And Prime Ideals

DEFINITION 3.1 : An ideal A of a ternary semigroup T is said to be a ***completely prime ideal*** of T provided $x, y, z \in$ T and $xyz \in$A implies either $x \in$ A or $y \in$A or $z \in$A.

EXAMPLE 3.2 : In the commutative ternary semigroup Z^- of all negative integers, the ideal P = { 3k : k ∈ Z^- } is a completely prime ideal. For x; y; $z \in Z^-$, xyz ∈ P ⇔ xyz is divisible by 3 ⇔ x is divisible by 3 or y is divisible by 3 or z is divisible by 3 ⇔ $x = 3k_1$ or $y = 3k_2$ or $z = 3k_3$ for $k_1;\ k_2;\ k_3 \in Z^-$ ⇔ $x \in P$ or $y \in P$ or $z \in P$.

EXAMPLE 3.3 : In example 3.2., P is a completely prime ideal. But the ideal Q = { 30k : k ∈ Z^-} is not a prime ideal of Z^-, since (-2) (-3) (-5) = -30 ∈ Q but (-2) $\notin$ Q, (-3) $\notin$ Q and (-5) $\notin$ Q.
THEOREM 3.4 : An ideal A of a ternary semigroup T is completely prime if and only if $x_1, x_2, \ldots, x_n$ ∈ T, *n* is odd natural number, $x_1 x_2 \ldots x_n \in A \Rightarrow x_i \in A$ for some *i* = 1, 2, 3, …..*n*.
Proof **:** Suppose that A is a completely prime ideal of T.
Let $x_1, x_2, \ldots, x_n$ ∈ T where n is odd natural number and $x_1 x_2 \ldots x_n$ ∈ A.
If n = 1 then clearly x_1 ∈ A.
If n = 3 then $x_1x_2x_3 \in A \Rightarrow x_1 \in A$ or $x_2 \in A$ or $x_3 \in A$.
If n = 5 then $x_1x_2x_3x_4x_5 \in A \Rightarrow x_1x_2x_3 \in A$ or $x_4 \in A$ or $x_5 \in A$
$\Rightarrow x_1 \in A$ or $x_2 \in A$ or $x_3 \in A$ or $x_4 \in A$ or $x_5 \in A$.
Therefore by induction of n is an odd natural number, then $x_1 x_2 \ldots x_n \in A$
$\Rightarrow x_i \in A$ for some i = 1, 2, 3, …..n.
The converse part is trivial.
THEOREM 3.5 : An ideal A of a ternary semigroup T is completely prime if and only if T\A is either subsemigroup of T or empty.
Proof : Suppose that A is a completely prime ideal of T and T\A $\neq \varnothing$.
Let $a, b, c \in$ T\A. Then $a \notin$A, $b \notin$A, $c \notin$A. Suppose if possible $abc \notin$ T\A.
Then $abc \in$A. Since A is completely prime, either $a \in$A or $b \in$A or $c \in$A.
It is a contradiction. Therefore $abc \in$T\A. Hence T\A is a subsemigroup of T.
Conversely suppose that T\A is a subsemigroup of T or T\A is empty.
If T\A is empty then A = T and hence A is completely prime.
Assume that T\A is a subsemigroup of T. Let $a, b, c \in$T and $abc \in$A.
Suppose if possible $a \notin$A, $b \notin$A, and $c \notin$A.
Then $a \in$T\A, $b \in$T\A and $c \in$T\A. Since T\A is a subsemigroup, $abc \in$ T\A and hence $abc \notin$ A. It is a contradiction. Hence either $a \in$A or $b \in$A or $c \in$A. Therefore A is a completely prime ideal of T.
DEFINITION 3.6 : An ideal A of a ternary semigroup T is said to be a ***prime ideal*** of T provided X,Y,Z are ideals of T and XYZ $\subseteq$ A $\Rightarrow$ X $\subseteq$A or Y $\subseteq$A or Z $\subseteq$A.
THEOREM 3.7 : In a ternary semigroup T, the following conditions are equivalent:
(i) A is a prime ideal of T.
(ii) For *a, b, c* ∈ T; < *a* > < *b* > < *c* > ⊆ A implies *a* ∈ A or *b* ∈ A or *c* ∈ A.
(iii) For *a*; *b*; *c* ∈ T; $T^1T^1aT^1T^1b\, T^1T^1c\, T^1T^1 \subseteq A$ implies *a* ∈ A or *b* ∈ A or *c* ∈ A.
Proof **:** (i) ⇒ (ii) : Suppose that A is a prime ideal of T. Then (i) ⇒ (ii) is obvious.
(ii) ⇒ (iii): Let a, b, c ∈ T such that $T^1T^1aT^1T^1b\, T^1T^1c\, T^1T^1 \subseteq A$.

Now $< a > < b > < c > = (T^1T^1aT^1T^1)(T^1T^1bT^1T^1)(T^1T^1cT^1T^1) \subseteq T^1T^1aT^1T^1b\ T^1T^1c\ T^1T^1 \subseteq A$
$\Rightarrow a \in A$ or $b \in A$ or $c \in A$.
(iii) ⇒ (i): Suppose that $a, b, c \in T$; $T^1T^1aT^1T^1bT^1T^1cT^1T^1 \subseteq A \Rightarrow a \in A$ or $b \in A$ or $c \in A$.
Let X, Y, Z be the three ideals of T and $XYZ \subseteq A$.
Suppose if possible $X \nsubseteq A$, $Y \nsubseteq A$, $Z \nsubseteq A$.
$X \nsubseteq A$, $Y \nsubseteq A$, $Z \nsubseteq A$, there exists a, b, c such that $a \in X$ and $a \notin A$, $b \in Y$ and $b \notin A$ and $c \in Z$ and $c \notin A$. $a \in X$, $b \in Y$, $c \in Z \Rightarrow abc \in XYZ \subseteq A$.
Now $T^1T^1aT^1T^1bT^1T^1cT^1T^1 \subseteq XYZ \subseteq A \Rightarrow a \in A$ or $b \in A$ or $c \in A$. It is a contradiction.
Therefore $X \subseteq A$ or $Y \subseteq A$ or $Z \subseteq A$ and hence A is a prime ideal of T.

THEOREM 3.8 : An ideal A of a ternary semigroup T is prime if and only if $X_1, X_2, \ldots, X_n \subseteq T$, n is odd natural number, $X_1 X_2 \ldots X_n \subseteq A \Rightarrow X_i \in A$ for some $i = 1, 2, 3, \ldots n$.

Proof **:** Suppose that A is a prime ideal of T.
Let $X_1, X_2, \ldots, X_n \subseteq T$, n is odd natural number and $X_1 X_2 \ldots X_n \subseteq A$
If $n = 1$ then clearly $X_1 \in A$.
If $n = 3$ then $X_1X_2X_3 \subseteq A \Rightarrow X_1 \subseteq A$ or $X_2 \subseteq A$ or $X_3 \subseteq A$.
If $n = 5$ then $X_1X_2X_3X_4X_5 \subseteq A \Rightarrow X_1X_2X_3 \in A$ or $X_4 \in A$ or $X_5 \in A$
$\Rightarrow X_1 \in A$ or $X_2 \in A$ or $X_3 \in A$ or $X_4 \in A$ or $X_5 \in A$.
Therefore by induction of n is an odd natural number, then $X_1 X_2 \ldots X_n \subseteq A$
$\Rightarrow X_i \subseteq A$ for some $i = 1, 2, 3, \ldots n$.
The converse part is trivial.

THEOREM 3.9 : Every completely prime ideal of a ternary semigroup T is a prime ideal of T.

Proof : Suppose that A is a completely prime ideal of a ternary semigroup T.
Let $a,b,c \in T$ and $< a > < b > < c > \subseteq A$. Then $abc \in A$. Since A is a completely prime , either $a \in A$ or $b \in A$ or $c \in A$. Therefore A is a prime ideal of T.
The following theorem is duo to Kar.S and Maity.B.K. [9].

THEOREM 3.10 : Let T be a commutative ternary semigroup . An ideal P of T is a prime ideal if and only if P is a completely prime ideal.

DEFINITION 3.11 : A nonempty subset A of a ternary semigroup T is said to be an ***m*-system** provided for any $a, b, c \in A$ implies that $T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \cap A \neq \emptyset$.

THEOREM 3.12 : An ideal A of a ternary semigroup T is a prime ideal of T if and only if T\A is an *m*-system of T or empty.

Proof : Suppose that A is a prime ideal of a ternary semigroup T and $T \backslash A \neq \emptyset$.
Let $a, b ,c \in T \backslash A$. Then $a \notin A$, $b \notin A$ and $c \notin A$.
Suppose if possible $T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \cap T \backslash A = \emptyset$.
$T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \cap T \backslash A = \emptyset \Rightarrow T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \subseteq A$.
Since A is prime, either $a \in A$ or $b \in A$ or $c \in A$.
It is a contradiction. Therefore $T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \cap T \backslash A \neq \emptyset$.
Hence T\A is an *m*-system.
Conversely suppose that T\A is either an *m*-system of T or $T \backslash A = \emptyset$.
If $T \backslash A = \emptyset$, then T = A and hence A is a prime ideal of T.
Assume that T\A is an *m*-system of T. Let $a, b, c \in T$ and $< a > < b > < c > \subseteq A$.
Suppose if possible $a \notin A$, $b \notin A$ and $c \notin A$. Then $a, b, c \in T \backslash A$. Sine T\A is an *m*-system,
$\Rightarrow T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \cap T \backslash A \neq \emptyset \Longrightarrow T^1T^1aT^1T^1bT^1T^1c\ T^1T^1 \nsubseteq A$
$\Longrightarrow <a><b><c> \nsubseteq A$. It is a contradiction.
Therefore $a \in A$ or $b \in A$ or $c \in A$. Hence A is a prime ideal of T.

DEFINITION 3. 13 : An ideal A of a ternary semigroup T is called a ***globally idempotent ideal*** if $A^n = A$ for all odd natural number n.

DEFINITION 3.14 : A ternary semigroup T is said to be a ***globally idempotent ternary semigroup*** if $T^n = T$ for all odd natural number n.

THEOREM 3.15 : If T is a globally idempotent ternary semigroup then every maximal ideal of T is a prime ideal of T.

Proof : Let M be a maximal ideal of T. Let A, B, C be three ideals of T such that
$ABC \subseteq M$. Suppose if possible $A \nsubseteq M$, $B \nsubseteq M$, $C \nsubseteq M$.
Now $A \nsubseteq M \Longrightarrow M \bigcup A$ is an ideal of T and $M \subset M \bigcup A \subseteq T$.
Since M is a maximal, $M \bigcup A = T$.
Similarly $B \nsubseteq M \Longrightarrow M \bigcup B = T$, $C \nsubseteq M \Longrightarrow M \bigcup C = T$.

Now T = TTT = (M $\bigcup$ A) (M $\bigcup$ B) (M $\bigcup$ C) $\subseteq$ M $\Rightarrow$ T $\subseteq$ M. Thus M = T.
It is a contradiction. Therefore either A $\subseteq$ M or B $\subseteq$ M or C $\subseteq$ M. Hence M is a prime.

DEFINITION 3.16 : An element a of a ternary semigroup T is said to be ***semisimple*** if n is odd natural number then $a \in < a >^n$ i.e. $< a >^n = < a >$.

DEFINITION 3.17 : A ternary semigroup T is called ***semisimple ternary semigroup*** provided every element in T is semisimple.

THEOREM 3.18 : If T is a globally idempotent ternary semigroup having maximal ideals then T contains semisimple elements.

***Proof* :** Suppose that T is a globally idempotent ternary semigroup having maximal ideals.
Let M be a maximal ideal of T. Then by theorem 3.15., M is prime.
Now if $a \in T \setminus M$ then $< a > \nsubseteq M$ and $< a >^n \nsubseteq M$. Then T = M ∪ < a > = M ∪ $< a >^n$.
Therefore $a \in < a >^n$ and hence $< a > = < a >^n$. Thus a is a semisimple element. Therefore T contains semisimple elements.

IV. Completely Semiprime Ideals And Semiprime Ideals

DEFINITION 4.1 : An ideal A of a ternary semigroup T is said to be a ***completely semiprime ideal*** provided $x \in T$, $x^n \in A$ for some odd natural number $n > 1$ implies $x \in A$.

EXAMPLE 4.2 : In commutative ternary semigroup Z^- of all negative integers, the ideal Q = {6k : $k \in Z^-$} is a semiprime ideal. For $x \in Z^-$, $x^3 \in Q \Leftrightarrow x^3$ is divisible by 6 $\Leftrightarrow x$ is divisible by 6 $\Leftrightarrow x = 6k_1$ for $k_1 \in Z^-$ $\Leftrightarrow x \in$ Q.

THEOREM 4.3 : An ideal A of a ternary semigroup T is completely semiprime if and only if $x \in T$, $x^3 \in A$ implies $x \in A$.

***Proof* :** Suppose that A is a completely semiprime ideal of T.
Then clearly $x \in T$, $x^3 \in A \Rightarrow x \in A$.
Conversely suppose that $x \in T$, $x^3 \in A \Rightarrow x \in A$.
We prove that $x \in T$, $x^n \in A$, for some odd natural number n > 1 $\Rightarrow x \in A \longrightarrow (1)$,
by induction on ***n*.** Clearly (1) is true for $n = 3$. Assume that (1) is true for $n = k$. i.e., $x^k \in A \Rightarrow x \in A$ for some odd natural number $k > 3$.
Suppose that $x^{k+2} \in A$. Then $x^{k+2} \in A \Rightarrow x^{k+2}.x^{k+2}.\ x^{k-4} \in A \Rightarrow x^{3k} \in A \Rightarrow (x^k)^3 \in A \Rightarrow x^k \in A \Rightarrow x \in A$. Therefore $x^k \in A \Rightarrow x \in A$.
By induction, $x^n \in A$ for some natural number n, $n > 1$ implies $x \in A$.
Therefore A is completely semiprime.

THEOREM 4.4 : If A is a completely semiprime ideal of a ternary semigroup T, then $x, y, z \in T$, $xyz \in A$ implies that xyTT$z \subseteq A$ and xTT$yz \subseteq A$.

***Proof* :** Let A be a completely semiprime ideal of a semigroup T. Let $x, y, z \in T$, $xyz \in A$.
Now $xyz \in A \Rightarrow (zxy)^3 = (zxy)(zxy)(zxy) = z(xyz)(xyz)xy \in A$.
$(zxy)^3 \in A$, A is completely semiprime implies $zxy \in A$.
Let $s, t \in T$. Consider $(xystz)^3 = (xystz)(xystz)(xystz) = xyst(zxy)st(zxy)sty \in A$.
$(xystz)^3 \in A$, A is completely semiprime implies $xystz \in A$.
Therefore $x, y, z \in T$, $xyz \in A \Rightarrow xystz \in A$ for all $s, t \in T \Rightarrow xyTTz \subseteq A$.
Now $xyz \in A \Rightarrow (yzx)^3 = (yzx)(yzx)(yzx) = yz(xyz)(xyz)x \in A$.
$(yzx)^3 \in A$, A is completely semiprime implies $\Rightarrow yzx \in A$.
Let $s, t \in T$. Consider $(xstyz)^3 = (xstyz)(xstyz)(xstyz) = xst(yzx)st(yzx)styz \in A$.
$(xstyz)^3 \in A$, A is completely semiprime implies $xstyz \in A$.
Therefore $x, y, z \in T$, $xstyz \in A$ for all $s, t \in T \Rightarrow x$TT$yz \subseteq A$.

COROLLARY 4.5 : If an ideal A of a ternary semigroup T is completely semiprime then $x, y, z \in T$, $xyz \in$ A $\Rightarrow < x >< y >< z > \subseteq$ A.

THEOREM 4.6 : Every completely prime ideal of a ternary semigroup T is a completely semiprime ideal of T.

Proof : Let A be a completely prime ideal of a ternary semigroup T. Suppose that
$x \in T$ and $x^3 \in A$. Since A is a completely prime ideal of T, $x \in A$.
Therefore T is a completely semiprime ideal.

THEOREM 4.7 : Let A be a prime ideal of a ternary semigroup T. If A is completely semiprime ideal of T then A is completely prime.

***Proof* :** Let $x, y, z \in T$ and $xyz \in A$. Since A is completely semiprime, by theorem 4.4., $xyz \in A \Rightarrow xyT^1T^1z \subseteq A$, $xT^1T^1yz \subseteq A \Rightarrow TxyTTzT \subseteq TAT \subseteq A \Rightarrow < x >< y >< z > \subseteq A$
$\Rightarrow x \in A$ or $y \in A$ or $z \in A$ and hence A is completely prime.

THEOREM 4.8 : The nonempty intersection of any family of a completely prime ideal of a ternary semigroup T is a completely semiprime ideal of T.

Proof : Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of a completely prime ideals of T such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \varnothing$.

It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is an ideal. Let $a \in$ T and $a^3 \in \bigcap_{\alpha\in\Delta} A_\alpha$. Then $a^3 \in A_\alpha$ for all $\alpha \in \Delta$.

Since A_α is completely prime, $a \in A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap_{\alpha\in\Delta} A_\alpha$.

Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a completely semiprime ideal of T.

DEFINITION 4.9 : Let T be a ternary semigroup. A non-empty subset A of T is said to be a ***d*-system** of T if $a \in$ A $\Rightarrow a^n \in$ A for all odd natural number *n*.

THEOREM 4.10 : An ideal A of a ternary semigroup T is completely semiprime if and only if T\A is a *d*-system of T or empty.

Proof : Suppose that A is a completely semiprime ideal of T and T\A $\neq \varnothing$.

Let $a \in$ T\A . Then $a \notin$ A. Suppose if possible $a^n \notin$ T\A for some odd natural number *n*.

Then $a^n \in$ A . Since A is a completely semiprime ideal then $a \in$ A.

It is a contradiction. Therefore $a^n \in$ T\A and hence T\A is a *d*-system .

Conversely suppose that T\A is a *d*-system of T or T\A is empty.

If T\A is empty then T = A and hence A is completely semiprime.

Assume that T\A is a *d*-system of T. Let $a \in$ T and $a^n \in$ A .

Suppose if possible $a \notin$ A. Then $a \in$ T\A.

Since T\A is a *d*-system, $a^n \in$ T\A. It is a contradiction. Hence $a \in$ A.

Thus A is a completely semiprime ideal of T .

DEFINITION 4.11 : An ideal A of a ternary semigroup T is said to be ***semiprime ideal*** provided X is an ideal of T and $X^n \subseteq$ A for some odd natural number *n* implies X ⊆ A.

THEOREM 4.12 : An ideal A of a ternary semigroup T is semiprime if and only if X is an ideal of T, $X^3 \subseteq$ A implies X ⊆ A.

Proof **:** Suppose that A is a semiprime ideal. Then clearly $X^3 \subseteq A \Rightarrow X \subseteq A$.

Conversely suppose that X is an ideal of T, $X^3 \subseteq A \Rightarrow X \subseteq A$.

We prove that $X^n \subseteq A$, for some odd natural number $n \Rightarrow X \subseteq A \longrightarrow (1)$, by induction on *n*. Since $X^3 \subseteq A \Rightarrow X \subseteq A$, (1) is true for *n* = 3.

Assume that $X^k \subseteq A$ for some odd natural number k, $1 \leq k < n \Rightarrow X \subseteq A$.

Now $X^{k+2} \subseteq A \Rightarrow X^{k+2}.X^{k+2}.X^{k-4} \subseteq A \Rightarrow X^{3k} \subseteq A \Rightarrow (X^k)^3 \subseteq A \Rightarrow X^k \subseteq A \Rightarrow X \subseteq A$ by assumption. By induction $X^n \subseteq A$ for some odd natural number $n \Rightarrow X \subseteq A$.

Therefore A is semiprime.

THEOREM 4.13 : Every prime ideal of a ternary semigroup is semiprime.

Proof **:** Suppose that A is a prime ideal of a ternary semigroup T. Let X be an ideal of T such that $X^3 \subseteq$ A. Since A is prime, X ⊆ A. Hence A is semiprime.

THEOREM 4.14 : If A is an ideal of a ternary semigroup T then the following are equivalent.

1. **A is a semiprime ideal.**
2. **For $a \in$ T; $< a >^3 \subseteq$ A implies $a \in$ A.**
3. **For $a \in$ T; $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq$ A implies $a \in$ A.**

Proof **:** (i) ⇒ (ii) : Suppose that A is a semiprime ideal of T. Then (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii): Let $a \in$ T such that $T^1T^1aT^1T^1a\, T^1T^1a\, T^1T^1 \subseteq$ A.

Now $< a >^3 = (T^1T^1aT^1T^1)(T^1T^1aT^1T^1)(T^1T^1aT^1T^1) \subseteq T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq A \Rightarrow a \in A$.

(iii) ⇒ (i): Suppose that $a \in$ T; $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq A \Rightarrow a \in$A.

Let X be the an ideals of T and $X^3 \subseteq$ A.

Suppose if possible X ⊈ A.

X ⊈ A there exists a such that $a \in$ X and $a \notin$ A. $a \in X \Rightarrow a^3 \in X^3 \subseteq$ A.

Now $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq X^3 \subseteq A \Rightarrow a \in$A. It is a contradiction.

Therefore X ⊆ A and hence A is a semiprime ideal of T.

THEOREM 4.15 : Every completely semiprime ideal of a ternary semigroup T is a semiprime ideal of T.
Proof : Suppose that A is a completely semiprime ideal of a ternary semigroup T.
Let $a \in$ T and $<a>^n \subseteq$ A for some odd natural number n.
Now $aaa.....a$(n odd terms) $\in < a^n > \subseteq <a>^n \subseteq$ A $\Rightarrow a^n \in$ A $\Rightarrow a \in$ A $\Rightarrow < a > \subseteq$ A.
Therefore A is a semiprime ideal of T.
THEOREM 4.16 : Let T be a commutative ternary semigroup. An ideal A of T is completely semiprime if and only if it is semiprime.
Proof : Suppose that A is a completely semiprime ideal of T. By theorem 4.14, A is a semiprime ideal of T.
Conversely suppose that A is a semiprime ideal of T.
Let $x \in$ T and $x^n \in$ A for some odd natural number n.
Now $x^n \in$ A $\Longrightarrow$ $< x >^n \subseteq$ A $\Longrightarrow$ $< x > \subseteq$ A $\Rightarrow x \in$ A. Since A is semiprime.
Therefore A is a completely semiprime ideal of T.
THEOREM 4.17 : The nonempty intersection of any family of prime ideals of a ternary semigroup T is a semiprime ideal of T.

Proof : Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a family of prime ideals of T such that $\bigcap_{\alpha\in\Delta} A_\alpha \neq \emptyset$. It is clear that $\bigcap_{\alpha\in\Delta} A_\alpha$ is an ideal. Let $a \in T$, $< a >^3 \subseteq \bigcap_{\alpha\in\Delta} A_\alpha$ then $< a >^3 \subseteq A_\alpha$ for all $\alpha \in \Delta$.
Since A_α is a prime, $< a > \subseteq A_\alpha$ for all $\alpha \in \Delta$ and hence $a \in A_\alpha$ for all $\alpha \in \Delta$.
So $a \in \bigcap_{\alpha\in\Delta} A_\alpha$. Therefore $\bigcap_{\alpha\in\Delta} A_\alpha$ is a semiprime ideal of T.
DEFINITION 4.18 : A non-empty subset A of a ternary semigroup T is said to be an ***n*-system** provided for any $a \in$ A implies that $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap A \neq \emptyset$.
THEOREM 4.19 : Every *m*-system in a ternary semigroup T is an *n*-system.
Proof : Let A be m-system of a ternary semigroup T. Let $a \in$ A. Since A is m-system, $a \in$ A, $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap A \neq \emptyset$. Therefore *A is* an n-system of T.
THEOREM 4.20 : An ideal Q of a ternary semigroup T is a semiprime ideal if and only if T\Q is an *n*-system of T (or) empty.
Proof : Suppose that A is a semiprime ideal of a ternary semigroup T and T\A $\neq \emptyset$.
Let $a \in$ T\A . Then $a \notin$ A.
Suppose if possible $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap T\backslash A = \emptyset$.
$T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap T\backslash A = \emptyset \Rightarrow T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq A$.
Since A is semiprime, either $a \in$ A.
It is a contradiction. Therefore $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap T\backslash A \neq \emptyset$.
Hence T\A is an n-system.
Conversely suppose that T\A is either an n-system or T\A = $\emptyset$.
If T\A = $\emptyset$ then T = A and hence A is a semiprime ideal.
Assume that T\A is an n-system of T. Let $a \in$ T and $< a > \subseteq$ A.
Let $a \in$ T\A, T\A is an n-system of T $\Rightarrow T^1T^1aT^1T^1aT^1T^1aT^1T^1 \cap T\backslash A \neq \emptyset$.
Suppose if possible $a \notin$ A. Then $a \in$ T\A. Since T\A is an m-system.
Then $T^1T^1aT^1T^1aT^1T^1aT^1T^1 \subseteq T\backslash A \Longrightarrow T^1T^1aT^1T^1aT^1T^1aT^1T^1 \nsubseteq A \Longrightarrow < a > \nsubseteq A$.
It is a contradiction. Therefore $a \in$ A. Hence A is a semiprime ideal of T.
THEOREM 4.21 : If N is an *n*-system in a ternary semigroup T and $a \in$ N, then there exist an *m*-system M in T such that $a \in$ M and M $\subseteq$ N.
Proof : We construct a subset M of N as follows:
Define $a_1 = a$, Since $a_1 \in$ N and N is an n-system, $(T^1T^1a_1T^1T^1a_1T^1T^1a_1T^1T^1) \cap N \neq \emptyset$.

Let $a_2 \in (T^1T^1a_1T^1T^1a_1T^1T^1a_1T^1T^1) \cap N$. Since $a_2 \in$ N and N is an n-system, $(T^1T^1a_2T^1T^1a_2T^1T^1a_2T^1T^1) \cap N \neq \emptyset$ and so on.
In general, if a_i has been defined with $a_i \in$ N, choose a_{i+1} as an element of $(T^1T^1a_2T^1T^1a_2T^1T^1a_2T^1T^1) \cap N$. Let M = $\{a_1, a_2.....a_i, a_{i+1}....\}$. Now $a \in$ M and M $\subseteq$ N.

We now show that M is an *m*-system.
Let $a_{i,} a_j$, $a_k \in$ M (for $i \leq j \leq k$).
Then $a_{k+1} \in T^1T^1a_kT^1T^1a_kT^1T^1a_kT^1T^1 \subseteq T^1T^1a_jT^1T^1a_jT^1T^1a_jT^1T^1$
$\subseteq T^1T^1a_iT^1T^1a_iT^1T^1a_iT^1T^1 \subseteq T^1T^1a_iT^1T^1a_jT^1T^1a_kT^1T^1$
$\Rightarrow a_{k+1} = T^1T^1a_iT^1T^1a_jT^1T^1a_kT^1T^1$. But $a_{k+1} \in$ M, so $a_{k+1} = T^1T^1a_iT^1T^1a_jT^1T^1a_kT^1T^1 \cap$ M,
Therefore M is an *m*-system.

V. Prime Radical And Completely Prime Radical

NOTATION 5.1 : If A is an ideal of a ternary semigroup T, then we associate the following four types of sets.
A_1 = The intersection of all completely prime ideals of T containing A.
A_2 = $\{x \in T: x^n \in A$ for some odd natural numbers $n\}$
A_3 = The intersection of all prime ideals of T containing A.
A_4 = $\{x \in T: < x >^n \subseteq$ A for some odd natural number $n\}$

THEOREM 5.2 : If A is an ideal of a ternary semigroup T, then A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

Proof : i) A $\subseteq A_4$: Let $x \in$ A . Then $< x > \subseteq$ A and hence $x \in A_4$
Therefore A $\subseteq A_4$
ii) $A_4 \subseteq A_3$: Let $x \in A_4$. Then $< x >^n \subseteq$ A for some odd natural number *n.*
Let P be any prime ideal of T containing A.
Then $< x >^n \subseteq$ A for some odd natural number $n \Rightarrow < x >^n \subseteq$ P.
Since P is prime , $< x > \subseteq$ P and hence $x \in$ P.
Since this is true for all prime ideals of P containing A, $x \in A_3$. Therefore $A_4 \subseteq A_3$
iii) $A_3 \subseteq A_2$: Let $x \in A_3$. Suppose if possible $x \notin A_2$.
Then $x^n \notin$ A for all odd natural number *n.*
Consider Q = $\bigcup x^n$ for all odd natural number *n*, and $x \in$ T.
Let *a*, *b*, *c* $\in$ Q . Then $a = (x)^r$, $b = (x)^s$, $c = (x)^t$ for some odd natural numbers *r, s, t*.
Therefore $abc = (x)^r\ (x)^s\ (x)^t = x^{r+s+t} \in Q$ and hence Q is a subsemigroup of T.
By theorem 3.5, P = T\Q is a completely prime ideal of T and $x \notin P$.
By theorem 3.9, P is a prime ideal of T and $x \notin P$. Therefore $x \notin A_3$. It is a contradiction.
Therefore $x \in A_2$ and hence $A_3 \subseteq A_2$.
iv) $A_2 \subseteq A_1$: Let $x \in A_2$. Now x $\in A_2 \Rightarrow x^n \in$ A for some odd natural number *n*.
Let P be any completely prime ideal of T containing A.
Then $x^n \in$ A $\subseteq$ P $\Rightarrow x^n \in$ P $\Rightarrow x \in$ P. Therefore $x \in A_1$. Therefore $A_2 \subseteq A_1$.
Hence A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$.

THEOREM 5.3 : A is an ideal of a commutative ternary semigroup T, then $A_1 = A_2 = A_3 = A_4$

Proof : By theorem5.2, A $\subseteq A_4 \subseteq A_3 \subseteq A_2 \subseteq A_1$. By theorem 3.10, in a commutative ternary semigroup T, an ideal A is a prime ideal if A is completely prime ideal.
So $A_1 = A_3$. By theorem 4.16, in a commutative ternary semigroup T an ideal A is semiprime if and only if A is completely semiprime ideal.
So $A_4 = A_2$ and hence $A_1 = A_2 = A_3 = A_4$.

NOTE 5.4 : In an arbitrary ternary semigroup $A_1 \neq A_2 \neq A_3 \neq A_4$.

EXAMPLE 5.5 : Let T be the free ternary semigroup generated by *a*, *b*, *c*.
It is clear that A = Ta^3T is an ideal of T. Since $a^5 \in$ Ta^3T, we have a$\in A_2$.

Evidently $(abc)^n \notin T a^3 T$ for all odd natural numbers *n* and thus abc $\notin A_2$.

Thus A_2 is not an ideal of T. Therefore $A_1 \neq A_2$ and $A_2 \neq A_3$**.**

DEFINITION 5.6 : If A is an ideal of a ternary semigroup T , then the intersection of all prime ideals of T containing A is called ***prime radical*** or simply ***radical*** of A and it is denoted by $\sqrt{A}$ or *rad* A.

DEFINITION 5.7: If A is an ideal of a ternary semigroup T , then the intersection of all completely prime ideals of T containing A is called ***completely prime radical*** or simply ***complete radical*** of A and it is denoted by *c.rad* A.

NOTE 5.8: If A is an ideal of a ternary semigroup T , then *rad* A = A_3 and *c.rad* A = A_1**.**

THEOREM 5.9: If $a \in \sqrt{A}$ **, then there exist a positive integer *n* such that** $a^n \in$ **A for some odd natural number** $n \in$ **N.**

Proof : By theorem 5.2, $A_3 \subseteq A_2$ and hence $a \in \sqrt{A} = A_3 \subseteq A_2$.

Therefore $a^n \in$ **A** for some odd natural number $n \in$ N.

THEOREM 5.10 : If A is an ideal of a commutative ternary semigroup T, then *rad* A = *c.rad* A.

proof : By theorem 5.3, *rad* A = *c.rad* A.

THEOREM 5.11 : If A is an ideal of a ternary semigroup T then *c.rad* A is a completely semiprime ideal of T.

proof : By theorem 4.6, *c.rad* A is a completely semiprime ideal of T.

THEOREM 5.12 : If A, B and C are any three ideals of a ternary semigroup T , then

i) **A** $\subseteq$ **B** $\Rightarrow \sqrt{A} \subseteq \sqrt{B}$

ii) if A ∩ B ∩ C ≠ ∅ then $\sqrt{ABC} = \sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$ **.**

proof : i) Suppose that $A \subseteq B$. If P is a prime ideal containing B then P is a prime ideal containing A. Therefore $\sqrt{A} \subseteq \sqrt{B}$.

ii) Let P be a prime ideal containing ABC. Then ABC ⊆ P $\Rightarrow$ A ⊆ P or B ⊆ P or C ⊆ P

$\Rightarrow A \cap B \cap C \subseteq$P. Therefore P is a prime ideal containing A $\cap$ B $\cap$ C.

Therefore *rad*($A \cap B \cap C$) ⊆ *rad*(ABC).

Now let P be a prime ideal containing $A \cap B \cap C$.

Then $A \cap B \cap C \subseteq$ P $\Rightarrow$ ABC ⊆ $A \cap B \cap C$ ⊆ P $\Rightarrow$ ABC ⊆ P.

Hence P is a prime ideal containing ABC. Therefore *rad* (ABC) ⊆ *rad*($A \cap B \cap C$).

Therefore *rad*(ABC) = *rad*($A \cap B \cap C$).

Since $A \cap B \cap C \neq \emptyset$, it is clear that A ∩ B is an ideal in T. Let $x \in \sqrt{A \cap B \cap C}$.

Then there exists a odd natural number $n \in$ N such that $x^n \in A \cap B \cap C$.

Therefore $x^n \in$ A, $x^n \in$ B and $x^n \in$ C. It follows that $x \in \sqrt{A}$, $x \in \sqrt{B}$ and $x \in \sqrt{C}$. Therefore $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$.

Consequently, $x \in \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ implies that there exists odd natural numbers *n, m, p* ∈ N such that $x^n \in$ A, $x^m \in$ B and $x^p \in$ C. Clearly, $x^{nmp} \in$ A ∩ B ∩ C.

Thus $x \in \sqrt{A \cap B \cap C}$. Therefore if A ∩ B ∩ C ≠ ∅ then $\sqrt{A \cap B \cap C} = \sqrt{A} \cap \sqrt{B} \cap \sqrt{C}$ **.**

iii) $\sqrt{A}$ = The intersection of all prime ideals of T containing A.

Now $\sqrt{\sqrt{A}}$ = The intersection of all prime ideals of T containing $\sqrt{A}$.

= The intersection of all prime ideals of T containing A = $\sqrt{A}$

Therefore $\sqrt{\sqrt{A}} = \sqrt{A}$.

THEOREM 5.13 : If A is an ideal of a ternary semigroup T then $\sqrt{A}$ **is a semiprime ideal of T.**

proof : By theorem 4.17 , $\sqrt{A}$ is a semiprime ideal of T.

THEOREM 5.14 : An ideal Q of ternary semigroup T is a semiprime ideal of T if and only if $\sqrt{Q}$ =Q.

Proof : Suppose that Q is a semiprime ideal. Clearly $Q \subseteq \sqrt{Q}$.
Suppose if possible $\sqrt{Q} \nsubseteq Q$.
Let $a \in \sqrt{Q}$ and $a \notin Q$. Now $a \notin Q \Rightarrow a \in S\backslash Q$ and Q is semiprime. By theorem 4.20,
S\Q is an *n*-system. By theorem 4.21, there exists an *m*-system M such that $a \in M \subseteq S\backslash Q$.
$Q \subseteq S\backslash M$ and now S\M is a prime ideal of S, $a \notin S\backslash M$. It is a contradiction.
Therefore $\sqrt{Q} \subseteq Q$. Hence $\sqrt{Q} = Q$.
Conversely suppose that Q is an ideal of S such that $\sqrt{Q} = Q$.
By corollary 5.13, $\sqrt{Q}$ is a semiprime ideal of S. Therefore Q is semiprime.

COROLLARY 5.15 : An ideal Q of a ternary semigroup T is a semiprime ideal if and only if Q is the intersection of all prime ideal of S contains Q.

***Proof* :** By theorem 5.14., Q is semiprime iff Q is the intersection of all prime ideals of T contains Q.

COROLLARY 5.16 : **If A is an ideal of a ternary semi group T, then A is the smallest semiprime ideal of T containing A.**

Proof : We have that $\sqrt{A}$ is the intersection of all prime ideals containing A in T.
Since intersection of prime ideals is semiprime, we have $\sqrt{A}$ is semiprime.
Further, let Q be any semiprime ideal containing A, i.e. $A \subseteq Q$. So $\sqrt{A} \subseteq \sqrt{Q}$.
Since Q is semiprime, By theorem 5.14, $\sqrt{Q} = Q$. Therefore $\sqrt{A} \subseteq Q$.
Hence $\sqrt{A}$ is the smallest semiprime ideal of S containing A.

THEOREM 5.16 : If P is a prime ideal of a ternary semigroup T, then $\sqrt{(P)^n}$ = *P* for all odd natural numbers *n* $\in$ N.

***Proof* :** We use induction on *n* to prove $\sqrt{P^n} = P$.

First we prove that $\sqrt{P} = P$. Since P is a prime ideal, $P \subseteq \sqrt{P} \subseteq P \Rightarrow \sqrt{P} = P$.

Assume that $\sqrt{P^k} = P$ for odd natural number k such that $1 \leq k < n$.

Now $\sqrt{P^{k+2}} = \sqrt{P^k.P.P} = \sqrt{P^k} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} \cap \sqrt{P} \cap \sqrt{P} = \sqrt{P} = P$.

Therefore $\sqrt{P^{k+2}} = P$. By induction $\sqrt{P^n}$ = P for all odd natural number $n \in$ N.

THEOREM 5.17: In a ternary semigroup T with identity there is a unique maximal ideal M such that $\sqrt{(M)^n}$ = M for all odd natural numbers *n* $\in$ N.

Proof: Since T contains identity, T is a globally idempotent ternary semigroup.
Since M is a maximal ideal of T, by theorem 3.15 M is prime.

By theorem 5.16, $\sqrt{(M)^n}$ = M for all odd natural numbers *n*.

Theorem 5.18: If A is an ideal of a ternary semigroup T then $\sqrt{A}$={*x* $\in$ T : every *m*-system of T containing *x* meets A } i.e., $\sqrt{A} = \{ x \in T : M(x) \cap A \neq \emptyset \}$.

Proof: Suppose that $x \in \sqrt{A}$. Let M be an m-system containing *x*.

Then T\M is a prime ideal of T and $x \notin$ T\M. If $M \cap A = \emptyset$ then $A \subseteq$ T\M.

Since T\M is a prime ideal containing A, $\sqrt{A} \subseteq$ T\M and hence $x \in$ T\M.

It is a contradiction. Therefore $M(x) \cap A \neq \emptyset$. Hence $x \in \{x \in T : M(x) \cap A \neq \emptyset\}$.

Conversely suppose that $x \in \{x \in T : M(x) \cap A \neq \emptyset\}$.

Suppose if possible $x \notin \sqrt{A}$. Then there exists a prime ideal P containing A such that $x \notin$ P.

Now T\P is an *m*-system and $x \in$ T\P. $A \subseteq P \Rightarrow T\backslash P \cap A = \emptyset \Rightarrow x \notin \{x \in T : M(x) \cap A \neq \emptyset\}$.

It is a contradiction. Therefore $x \in \sqrt{A}$. Thus $\sqrt{A} = \{x \in T : M(x) \cap A \neq \emptyset\}$.

VI. Conclusion

Anjaneyulu. A initiated the study of pseudo symmetric ideals in semigroups, Madhusudhana Rao. D, Anjaneyulu. A. and Gangadhara Rao. A. initiated the the study of theory of Γ-ideals in Γ-semigroups and V. B.

Subrahmanyeswara Rao Seetamraju, Anjaneyulu and Madhusudhana Rao initiated the study of theory of ideals in partially ordered Γ-semigroups and hence the study of ideals in semigroups, Γ-semigroups and partially ordered Γ-semigorups creates a platform for the ideals in ternary semigroups.

Acknowledgements

We thank the Department of Mathematics, VSR & NVR College, Tenlai, A.P. India for giving their valuable support in preparing this paper.

References

[1] **Anjaneyulu. A and Ramakotaiah. D**., on a class of semigroups , Simon – Stivin, vol .34(1980), 241-249.
[2] **Anjaneyulu. A .**, Structure and ideal theory of Duo semigroups, Semigroup forum, vol .22(1981), 237-276.
[3] **Clifford A.H and Preston G.B**., The algebroic theory of semigroups , vol – I American Math. Society, Province (1961).
[4] **Clifford A.H and Preston G.B**., The algebroic theory of semigroups , vol – II American Math. Society, Province (1967).
[5] Dutta.T.K., **Kar.S abd Maity.B.K**., On Ideals of regular ternary semigroups , Internal. J. Math. Math. Sci. 18 (1993), 301-308.
[6] Hewitt. **E. and Zuckerman H.S**., Ternary opertions and semigroups, semigroups, Proc. Sympos. Wayne State Univ., Detroit, 1968, 33-83.
[7] **Iampan . A**., Lateral ideals of ternary semigroups , Ukrainian Math, Bull., 4 (2007), 323-334.
[8] **Kar.S** ., On ideals in ternary semigroups . Int. J. Math. Gen. Sci., 18 (2003) 3013- 3023.
[9] **Kar.S and Maity.B.K**., *Some ideals of ternary semigroups* . Analele Stintifice Ale Universitath "ALI CUZA" DIN IASI(S.N) Mathematica, Tumul LVII. 2011-12.
[10] **Kasner . E**., An extension of the group concept , Bull. Amer. Math. Society, 10 (1904), 290-291.
[11] **Kim Ki.H and Roush F.W**., Ternary semigroup on each pair of factors, Simon Stevin 34(1980),No.2,63-74.
[12] **Kuroki. N**., Rough ideals in semigroups, Information sciences, Vol.100 (1997), 139-163.
[13] **Lehmer . D.H**., A ternary analave of abelian groups , Amer. J. Math., 39 (1932), 329-338.
[14] **Lyapin.E.S**., Realisation of ternary semigroup, Russian Modern Algebra, Leningrad University, Leningrad, 1981, pp. 43-48.
[15] **Petrch.M**., Introduction to semigroups , Merril publishing company, Columbus, Ohio (1973).
[16] **Santiago. M. L**. and Bala S.S., " Ternary semigroups" Semigroups Forum,Vol. 81,no. 2, pp. 380-388, 2010.
[17] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao. D.,** *On ternary semigroups*, International eJournal of Mathematics, Engineering and Technology accepted for publication.
[18] **Sarala. Y, Anjaneyulu. A** and **Madhusudhana Rao. D.,** *Ideals in ternary semigroups*, submitted for publication in International eJournal of Mathematics, Engineering and Technology accepted for publication.
[19] **Shabir.M and Basher.S.**, Prime Ideals in ternary semigroups Asian European J. Math. 2 (2009), 139-132.
[20] **Sioson. F. M**., Ideal theory in ternary semigroups , Math. Japan., 10 (1963) , 63-84.