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# On Complete lift and Nijenhuis tensor of (1,1) tensorfield of the Basespace in the Cotangent Bundle

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***Abstract:*** *If M is a differtiable manifold of dimension n,then its cotangent bundle $T^*(M)$ is a differtiable manifold of dimension 2n[1].In the present paper, complete and horizontal lifts of (1,1) tensor fields of M ,which are tensor fields of same type in $T^*(M)$ , are studied. The Nijenhuis tensor of complete lift and Integrability of the Hsu-structure in$T^*(M)$ are also studied.*
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## I. Introduction

Let M be a differentiable manifold of class $C^\infty$ anddimension n. At each point P of M , there is associated an n-dimensional vector space of tangent vectors called tangent space,denoted by $T_P(M)$. If $T_P{}^*(M)$ be dual space of $T_P(M)$. We denote$\cup_{P\in M} T_P{}^*(M) = T^*(M)$, and call $T^*(M)$ the cotangent bundle of M. It can be shown that $T_P{}^*(M)$ is also a differentiable manifold of dimension 2n.Let $\pi$ be projection map $T^*(M) \longrightarrow M$.Let U be the coordinate neighborhood of P in M with coordinate functions $(x^1, x^2, \ldots x^n)$or $(x^h)$ .Then $\pi^{-1}(U)$ is open subset in$T^*(M)$ with coordinate functions$(x^h, p_i)$, $h, i = 1, 2 \ldots\ldots n$, and $p_i$are components of 1-form at P.Let U and $U'$ be the two coordinate neighborhoods in M such that $U \cap U' \neq \emptyset$, then $\pi^{-1}(U)$ and $\pi^{-1}(U')$ are open subsets in $T^*(M)$ and intersect each other.The local coordinate systems $(x^h)$ and $(x^{h'})$ in U , $U'$ respectively induce local coordinate systems $(x^h, p_i)$ and $(x^{h'} p_i')$ in $\pi^{-1}(U)$ and $\pi^{-1}(U')$ respectively.In the intersecting region $\pi^{-1}(U) \cap \pi^{-1}(U')$ , we have the law of transformation

$$\text{(i) } x^{h'} = x^{h'}(x^h) \quad \text{(ii) } p_{i'} = \frac{\partial x^i}{\partial x^{i'}} p_i \qquad \ldots\ \ldots\ldots \ (1.1)$$

We call M as the base space. Suppose M admits a tensorfield F of type (1 ,1). Then its Complete lift $F^C$ is a (1,1) tensorfield in $T^*(M)$ with local components [1]

$$F^c = \begin{bmatrix} F_i{}^h & 0 \\ p_a\left(\frac{\partial F_h^a}{\partial x^i} - \frac{\partial F_i{}^a}{\partial x^h}\right) & F_h{}^i \end{bmatrix} \qquad (1.2)$$

Where $(x^1, x^2, \ldots x^n)$ is local coordinate system in U and $F_i{}^h$ are components of (1,1) tensorfield F in M.
Suppose $\nabla$ is a symmetric affine in M with local components $\Gamma_{ji}^h$ in U. If $\Gamma_{ji} = p_a \Gamma_{ji}^h$ , the horizontal lift $F^H$ of F is a (1,1) tensorfield in $T^*(M)$ defined as [1]

$$F^c = \begin{bmatrix} F_i{}^h & 0 \\ -\Gamma_{ia} F_h^a + \Gamma_{ih} F_i^a & F_h{}^i \end{bmatrix} \qquad (1.3)$$

The Nijenhuis tensor N(X ,Y) of (1,1) tensorfield F in M is a (1,2) tensor given by

$$N(X,Y) = [FX , FY]-F [ FX ,Y]-F[ X ,FY ]+F^2[ X ,Y ] \qquad (1.4)$$

The structure is called integrable if its Nijenhuis tensor vanishes.

## II. Hsu Structure in $T^*(M)$

Suppose that the base space admits a (1,1) tensorfield F satisfying

$$F^{10} + \lambda^r F^6 + \mu^r F^2 = 0 \qquad (2.1)$$

Where $\lambda, \mu$ are scalars. Let us call that M admits $F_{\lambda,\mu}(10, -4)$ Hsu Structure [5]
The Complete lift $F^C$ is a (1 ,1)tensorfield in $T^*(M)$ with local components given by the equation (1.2). If we put [4] $p_a\left(\frac{\partial F_h^a}{\partial x^i} - \frac{\partial F_i{}^a}{\partial x^h}\right) = 2p_a\,\partial\,[iF_h^a]$

(2.2)

Then we have $(F^c) = \begin{bmatrix} F_i^{\ h} & 0 \\ 2p_a\, \partial\, [iF_h^a] & F_h^i \end{bmatrix}$

(2.3)

$$(F^c)^2 = \begin{bmatrix} F_i^{\ h} F_j^{\ i} & 0 \\ 2p_a\, \partial[iF_h^a]F_j^i + 2p_t\, \partial[iF_i^t]F_h^i & F_i^{\ j} F_h^{\ i} \end{bmatrix}$$

If we put

$$2p_a\, \partial[iF_h^a]F_j^i + 2p_t\, \partial[iF_i^t]F_h^i = L_{hj}$$

then $$(F^c)^2 = \begin{bmatrix} F_i^{\ h} F_j^{\ i} & 0 \\ L_{hj} & F_i^{\ j} F_h^{\ i} \end{bmatrix} \tag{2.4}$$

Similarly

$$(F^c)^4 = \begin{bmatrix} F_i^{\ h} F_j^{\ i} F_k^{\ j} F_l^{\ k} & 0 \\ F_k^{\ j} F_l^{\ k} L_{hj} + F_i^{\ j} F_h^{\ i} L_{jl} & F_k^{\ l} F_j^{\ k} F_i^{\ j} F_h^{\ i} \end{bmatrix}$$

Putting $F_k^{\ j} F_l^{\ k} L_{hj} + F_i^{\ j} F_h^{\ i} L_{jl} = L_{hl}$

$$(F^c)^4 = \begin{bmatrix} F_i^{\ h} F_j^{\ i} F_k^{\ j} F_l^k & 0 \\ L_{hl} & F_k^{\ l} F_j^{\ k} F_i^{\ j} F_h^{\ i} \end{bmatrix} \tag{2.5}$$

Again putting $F_m^l F_n^m L_{hl} + F_k^{\ l} F_j^{\ k} F_i^{\ j} F_h^{\ i} L_{ln} = L_{hn}$ and proceeding in the similar way , we have

$$(F^c)^6 = \begin{bmatrix} F_i^{\ h} F_j^{\ i} F_k^{\ j} F_l^{\ k} F_m^l F_n^m & 0 \\ L_{hn} & F_m^n F_l^m F_k^l F_j^{\ k} F_i^{\ j} F_h^i \end{bmatrix} \tag{2.6}$$

In the same way, we have

$$(F^c)^{10} = \begin{bmatrix} F_i^{\ h} F_j^{\ i} F_k^{\ j} F_l^{\ k} F_m^l F_n^m F_p^{\ n} F_q^{\ p} F_r^q F_s^r & 0 \\ L_{hs} & F_r^{\ s} F_q^{\ r} F_p^{\ q} F_n^{\ p} F_m^n F_l^m F_k^{\ l} F_j^{\ k} F_i^j F_h^i \end{bmatrix} \tag{2.7}$$

where $F_p^{\ n} F_q^{\ p} F_r^{\ q} F_s^r L_{hn} + F_m^n F_l^m F_k^{\ l} F_j^{\ k} F_i^{\ j} F_h^{\ i} L_{hl} = L_{hs}$.

Thus in $T^*(M)$

$$(F^c)^{10} + \lambda^r (F^c)^6 + \mu^r (F^c)^2 = 0 \text{ holds if and only if}$$
$$L_{hs} + \lambda^r L_{hn} + \mu^r L_{hj} = 0$$

Hence , we have the following theorem:

**Theorem (2.1)** : In order that the complete lift of (1,1) tensorfield F admitting $F_{\lambda,\mu}(10\ ,-4)$ Hsu-structure in M may have the similar structure in $T^*(M)$ , it is necessary and sufficient that $L_{hs} + \lambda^r L_{hn} + \mu^r L_{hj} = 0$

## III. The Nijenhuis Tensor

Since the base space M admits $F_{\lambda,\mu}(10\ ,-4)$ Hsu-structure , the Nijenhuis Tensor of complete lift of $F^{10}$ in $T^*(M)$ is given by

$$\begin{aligned} N_{(F^{10})^C,\ (F^{10})^C}(X^C, Y^C) = {} & [(\lambda^r F^6 + \mu^r F^2)^C X^C, (\lambda^r F^6 + \mu^r F^2)^C Y^C] \\ & -(\lambda^r F^6 + \mu^r F^2)^C [(\lambda^r F^6 + \mu^r F^2)^C X^C, Y^C] \\ & -(\lambda^r F^6 + \mu^r F^2)^C [X^C, (\lambda^r F^6 + \mu^r F^2)^C Y^C] \\ & +(\lambda^r F^6 + \mu^r F^2)^C (\lambda^r F^6 + \mu^r F^2)^C [X^C, Y^C] \end{aligned}$$

In view of [1] ( pp 243)

$(\lambda^r F^6 + \mu^r F^2)^C X^C = ((\lambda^r F^6 + \mu^r F^2) X)^C + \gamma\, (L_X (\lambda^r F^6 + \mu^r F^2))$

$L_X$ denotes the Lie derivative via X and $\gamma(T)$ is a tensor field of type ( r , s-1) in $T^*(M)$ for a tensor field T of type ( r , s) in M. If we further assume that $\gamma\, (L_X (\lambda^r F^6 + \mu^r F^2)) = 0$ etc , we have

$$N_{(F^{10})^C,\ (F^{10})^C}(X^C, Y^C) = [((\lambda^r F^6 + \mu^r F^2)^C X)^C, ((\lambda^r F^6 + \mu^r F^2)^C Y)^C]$$

$$-(\lambda^r F^6 + \mu^r F^2)^C [((\lambda^r F^6 + \mu^r F^2)^C X)^C, Y^C]$$
$$-(\lambda^r F^6 + \mu^r F^2)^C [X^C, ((\lambda^r F^6 + \mu^r F^2)^C Y)^C]$$
$$+(\lambda^r F^6 + \mu^r F^2)^C (\lambda^r F^6 + \mu^r F^2)^C [X^C, Y^C]$$

Further , suppose that

$N_{(F^m)^C, (F^n)^C}(X^C, Y^C) = 0$ for $m \neq n$ and

Since $(F^6 X)^C = (F^6) X^C$ as $\gamma(L_X F^6) = 0$ etc , we arrive after simplification at the result

$$N_{(F^{10})^C, (F^{10})^C}(X^C, Y^C) = \lambda^{2r} N_{(F^6)^C, (F^6)^C}(X^C, Y^C) + \mu^{2r} N_{(F^2)^C, (F^2)^C}(X^C, Y^C) \quad (3.1)$$

Thus, we have the following theorem.

**Theorem (3.1**) : For (1 ,1) tensorfield F on the base space M admitting $F_{\lambda,\mu}(10, -4)$ Hsu-structure , the Nijenhuistersors of $(F^{10})^C$ ,$(F^6)^C$ and $(F^2)^C$ in $T^*(M)$ are connected by the equation (3.1) provided the Lie derivatives X of various powers of F vanish and $N_{(F^m)^C, (F^n)^C}(X^C, Y^C) = 0$ for $m \neq n$.
Consequently in the cotangent bundle $T^*(M)$ , the Hsu-structure induced by$(F^{10})^C$ will be integrableiff the Hsu-structures induced by $(F^6)^C$ and$(F^2)^C$ are integrable .

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