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Using Stirling's Interpolation to Find Gauss and Mean Curvature for the Surface

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Abstract: *In this paper, we can use Stirling's interpolation to compute the mean and gauss curvatures for any surface z = f(x, y) at any point (x_0, y_0).*
Keywords: *Differential geometry, Gauss curvature, Mathematica program, Mean curvature, Stirling's interpolation.*

I. Introduction

Surface characteristics can be divided into two types:

1- Local characteristics are associated with points on the surface, and can be discovered by examining the local neighborhoods of points.

It is including:

- continuity
- mean curvature
- gaussian curvature
- singularities
- critical points: minima, maxima, and saddle points

2- Global characteristics are associated with the surface as a whole and cannot be determined strictly by looking at local neighborhoods

It is including:

- embeddedness
- orientability
- symmetry, including periodicity
- genus
- ends
- total curvature
- skeletal graphs

Definition 1.1.The gaussian curvature [1] of a surface at a point is the product of the principal curvatures K = k_1 k_2 at that point. The tangent plane of any point with positive gaussian curvature touches the surface at a single point, whereas the tangent plane of any point with negative gaussian curvature cuts the surface.

Definition 1.2.The mean curvature [1] half the sum of the principal curvatures H = (k_1 + k_2)/2, and any point with zero mean curvature has negative or zero gaussian curvature. Surfaces with zero mean curvature everywhere are minimal surfaces.

The gaussian and mean curvature play a very important role in the theory of surfaces and these are defined at each point on the surface.

Definition 1.3.The real-valued function of two variables z = f(x, y) is a surface in $\Re^3$ [2].

II. The Formulas of Gauss and Mean curvatures

Let k_1, k_2 $(k_1 \geq k_2)$ denote the two principle curvature, the Gaussian curvature K and the mean curvature H are defined as K = k_1 k_2 and H = $\frac{1}{2}(k_1 + k_2)$. Formulas for computing K and H [3] are

$$\text{Mean curvature} = H = \frac{(1+f_x^2)f_{yy}+(1+f_y^2)f_{xx}-2f_xf_yf_{xy}}{2(1+f_x^2+f_y^2)^{\frac{3}{2}}}, \qquad (1)$$

$$\text{Gauss curvature} = K = \frac{f_{xx}+f_{yy}-f_{xy}^2}{(1+f_x^2+f_y^2)^2}. \qquad (2)$$

Therefore the principle curvature k_1, k_2 are

$$\mathbf{k_1 = H + \sqrt{H^2-K},\ k_2 = H - \sqrt{H^2-K}.} \qquad (3)$$

We shall denote partial derivatives of functions by

$$f_x = \frac{\partial f}{\partial x},\ f_{xx} = \frac{\partial^2 f}{\partial x^2},\ f_y = \frac{\partial f}{\partial y},\ f_{yy} = \frac{\partial^2 f}{\partial y^2},\ f_{xy} = \frac{\partial^2 f}{\partial x \partial y}. \qquad (4)$$

III. Using Mathematica Program

From Mathematica program we can compute:

1.1. Expand (a+b)$^{\mathbf{n}}$ from n=1 , 2, 3, 4, 5, 6, 7, 8, 9, 10.

This is the program to evaluate the expand

```
Text@Style[Grid[{#, "=", Expand[#]} & /@Table[(a - b)^n, {n, 10}], Alignment → Left], "TraditionalForm"]
```

$a-b = a-b$

$(a-b)^2 = a^2-2ab+b^2$

$(a-b)^3 = a^3-3a^2b+3ab^2-b^3$

$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4$

$(a-b)^5 = a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5$

$(a-b)^6 = a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6$

$(a-b)^7 = a^7-7a^6b+21a^5b^2-35a^4b^3+35a^3b^4-21a^2b^5+7ab^6-b^7$

$(a-b)^8 = a^8-8a^7b+28a^6b^2-56a^5b^3+70a^4b^4-56a^3b^5+28a^2b^6-8ab^7+b^8$

$(a-b)^9 = a^9-9a^8b+36a^7b^2-84a^6b^3+126a^5b^4-126a^4b^5+84a^3b^6-36a^2b^7+9ab^8-b^9$

$(a-b)^{10} = a^{10}-10a^9b+45a^8b^2-120a^7b^3+210a^6b^4-252a^5b^5+210a^4b^6-120a^3b^7+45a^2b^8-10ab^9+b^{10}$

1.2. We use Do to write a procedural program for calculating the factorial of any integer **by** using a command fac to calculate the factorial of any positive integer n and SetDelayed (:=) for we do not want to evaluate the right side until fac is called.

```
fac[n_] := (x = 1; Do[x = k * x, {k, n}]; x)

TabView[Table[ToString[n] "!" → fac[n], {n, 10}], 10]
```

!1	!2	!3	!4	!5	!6	!7	!8	!9	!10

3 628 800

Or use this easy command:

```
Grid[Range[10] /. n_Integer → {Defer[n!], "=", n!}, Alignment → Right]
```

1!	=	1
2!	=	2
3!	=	6
4!	=	24
5!	=	120
6!	=	720
7!	=	5040
8!	=	40 320
9!	=	362 880
10!	=	3 628 800

IV. Stirling's Central-Difference Interpolation

Definition 4.1. Stirling's Central-Difference Interpolation formula is based on a diagonal difference table rather than a horizontal difference table.

Without going in to the distinctions between the two types of difference tables, we can write down an interpolation formula including only second differences that is similar to Newton's interpolation formulas

Lemma 4.2. [4] Stirling's interpolation formula takes the form: [http://eom.springer.de/s/s087840.htm]

$$f_p(x) = f_p(x_0 + ph) = f_0 + p f_0^1 + \frac{p^2}{2!} f_0^2 + \dots\dots\dots + \frac{p(p^2-1)\dots.[p^2-(n-1)^2]}{(2n-1)!} f_0^{2n-1} + \frac{p(p^2-1)\dots.[p^2-(n-1)^2]}{2n!} f_0^{2n}. \qquad (5)$$

Where $p = (x - x_n) / h = (x - x_n) / (x_{n+1} - x_n)$ (6)

Since $f_p(x) = f(x_0 + ph) = f(x_0 + p\Delta x).$ (7)

Note that $f(x_0) = f_0,\ f(x_1) = f_1, \dots.,\ f(x_n) = f_n.$ (8)

Remark 4.3. For small p, Stirling's interpolation formula is more exact than other interpolation formulas.

Definition 4.4. [4] The formula for any central difference δ can be

$$f_{n+1/2}^{2m+1} = f_{n+1}^{2m} - f_n^{2m},$$
$$f_n^{2m} = f_{n+1/2}^{2m-1} - f_{n-1/2}^{2m-1}. \quad (m = 0, 1, 2, \dots, n = \dots, -1, 0, 1, \dots\dots) \qquad (9)$$

We can write these equations as follows:

$$\delta^{2m} f_n = \delta^{2m-1} f_{n+1/2} - \delta^{2m-1} f_{n-1/2},$$
$$\delta^{2m+1} f_{n+1/2} = \delta^{2m} f_{n+1} - \delta^{2m} f_n, \qquad (10)$$
$$\delta^{2m+1} f_{n-1/2} = \delta^{2m} f_n - \delta^{2m} f_{n-1}.$$

So $f_0^1 = \delta f_0,\ f_0^2 = \delta^2 f_0,\ f_0^3 = \delta^3 f,\ f_n^m = \delta^m f_n$ and so on.

Then

$$\delta^2 f_0 = \delta f_{1/2} - \delta f_{-1/2} = f_1 - f_0 - f_0 + f_{-1} = f_1 - 2f_0 + f_{-1}. \qquad (11)$$

$$\delta^4 f_0 = \delta^3 f_{1/2} - \delta^3 f_{-1/2} = \delta^2 f_1 - \delta^2 f_0 - \delta^2 f_0 + \delta^2 f_{-1} = f_2 - 2f_1 + f_0 - 2f_1 + 4f_0 - 2f_{-1} + f_0 - 2f_{-1} + f_{-2}.$$

$$\therefore \delta^4 f_0 = f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}. \qquad (12)$$

$$\delta^6 f_0 = \delta^5 f_{1/2} - \delta^5 f_{-1/2} = \delta^4 f_1 - \delta^4 f_0 - \delta^4 f_0 + \delta^4 f_{-1} = f_3 - 4f_2 + 6f_1 - 4f_0 + f_{-1} - 2f_2 + 8f_1 - 12f_0 + 8f_{-1} - 2f_{-2} + f_1 - 4f_0 + 6f_{-1} - 4f_{-2} + f_{-3}.$$

$$\therefore \delta^6 f_0 = f_3 - 6f_2 + 15f_1 - 20f_0 + 15f_{-1} + -6f_{-2} + f_{-3}. \qquad (13)$$

$$\delta^8 f_0 = \delta^7 f_{1/2} - \delta^7 f_{-1/2} = \delta^6 f_1 - \delta^6 f_0 - \delta^6 f_0 + \delta^6 f_{-1} = f_4 - 6f_3 + 15f_2 - 20f_1 + 15f_0 - 6f_{-1} + f_{-2} - 2f_3 + 12f_2 - 30f_1 + 40f_0 - 30f_{-1} + 12f_{-2} - 2f_{-3} + f_2 - 6f_1 + 15f_0 - 20f_{-1} + 15f_{-2} - 6f_{-3} + f_{-4}.$$

$$\therefore \delta^8 f_0 = f_4 - 8f_3 + 28f_2 - 56f_1 + 70f_0 - 56f_{-1} + 28f_{-2} - 8f_{-3} + f_{-4}. \qquad (14)$$

$$\delta^{10} f_0 = \delta^9 f_{1/2} - \delta^9 f_{-1/2} = \delta^8 f_1 - \delta^8 f_0 - \delta^8 f_0 + \delta^8 f_{-1} = f_5 - 8f_4 + 28f_3 - 56f_2 + 70f_1 - 56f_0 + 28f_{-1} - 8f_{-2} + f_{-3} - 2f_4 + 16f_3 - 56f_2 + 112f_1 - 140f_0 + 112f_{-1} - 56f_{-2} + 16f_{-3} - 2f_{-4} + f_3 - 8f_2 + 28f_1 - 56f_0 + 70f_{-1} - 56f_{-2} + 28f_{-3} - 8f_{-4} + f_{-5}.$$

$$\therefore \delta^{10} f_0 = f_5 - 10f_4 + 45f_3 - 120f_2 + 210f_1 - 252f_0 + 210f_{-1} - 120f_{-2} + 45f_{-3} - 10f_{-4} + f_{-5}. \qquad (15)$$

From [5]

$$f_n^{2k-1} = \mu\left[f_{n+1/2}^{2k-1} + f_{n-1/2}^{2k-1}\right] = \frac{1}{2}\left[f_{1/2}^{2k-1} + f_{-1/2}^{2k-1}\right], \ k = 1, 2, 3, \dots\dots, n = \dots, -1, 0, 1, \dots\dots$$

$$f_{n+1/2}^{2k-1} = f_{n+1}^{2k} - f_{n}^{2k} \tag{16}$$

Where μ [6] is averaging or mean operator defined by $\mu y_n = \frac{1}{2}\left[y_{n+\frac{1}{2}} + y_{n-\frac{1}{2}} \right]$ (17)

We can write these equations as follows:

$$\begin{aligned} \delta^{2k-1} f_n &= \mu[\delta^{2k-1} f_{n+1/2} - \delta^{2k-1} f_{n-1/2}] = \frac{1}{2}[\delta^{2k-1} f_{n+1/2} - \delta^{2k-1} f_{n-1/2}], \\ \delta^{2k-1} f_{n+1/2} &= \delta^{2k-2} f_{n+1} - \delta^{2k-2} f_n, \\ \delta^{2k-1} f_{n-1/2} &= \delta^{2k-2} f_n - \delta^{2k-2} f_{n-1}. \end{aligned} \tag{18}$$

Then

$$\delta f_0 = \frac{1}{2}[f_{1/2} + f_{-1/2}] = \frac{1}{2}[f_1 - f_0 + f_0 - f_{-1}] = \frac{1}{2}[f_1 - f_{-1}]. \tag{19}$$

$$\begin{aligned} \delta^3 f_0 &= \frac{1}{2}[\delta^3 f_{1/2} + \delta^3 f_{-1/2}], \\ \because \delta^3 f_{1/2} &= \delta^2 f_1 - \delta^2 f_0 = f_2 - 2f_1 + f_0 - f_1 + 2f_0 - f_{-1} = f_2 - 3f_1 + 3f_0 - f_{-1}, \\ \because \delta^3 f_{-1/2} &= \delta^2 f_0 - \delta^2 f_{-1} = f_1 - 2f_0 + f_{-1} - f_0 + 2f_{-1} - f_{-2} = f_1 - 3f_0 + 3f_{-1} - f_{-2}, \\ \therefore \delta^3 f_0 &= \frac{1}{2}[f_2 - 2f_1 + 2f_{-1} - f_{-2}]. \end{aligned} \tag{20}$$

$$\begin{aligned} \delta^5 f_0 &= \frac{1}{2}[\delta^5 f_{1/2} + \delta^5 f_{-1/2}], \\ \because \delta^5 f_{1/2} &= \delta^4 f_1 - \delta^4 f_0 = f_3 - 4f_2 + 6f_1 - 4f_0 + f_{-1} - f_2 + 4f_1 - 6f_0 + 4f_{-1} - f_{-2} \\ &= f_3 - 5f_2 + 10f_1 - 10f_0 + 5f_{-1} - f_{-2}, \\ \because \delta^5 f_{-1/2} &= \delta^4 f_0 - \delta^4 f_{-1} = f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2} - f_1 + 4f_0 - 6f_{-1} + 4f_{-2} - f_{-3} \\ &= f_2 - 5f_1 + 10f_0 - 10f_{-1} + 5f_{-2} - f_{-3}, \\ \therefore \delta^5 f_0 &= \frac{1}{2}[f_3 - 4f_2 + 5f_1 - 5f_{-1} + 4f_{-2} - f_{-3}]. \end{aligned} \tag{21}$$

$$\begin{aligned} \delta^7 f_0 &= \frac{1}{2}[\delta^7 f_{1/2} + \delta^7 f_{-1/2}], \\ \because \delta^7 f_{1/2} &= \delta^6 f_1 - \delta^6 f_0 = f_4 - 6f_3 + 15f_2 - 20f_1 + 15f_0 - 6f_{-1} + f_{-2} - \\ &\quad - f_3 + 6f_2 - 15f_1 + 20f_0 - 15f_{-1} + 6f_{-2} - f_{-3} \\ &= f_4 - 7f_3 + 21f_2 - 35f_1 + 35f_0 - 21f_{-1} + 7f_{-2} - f_{-3}, \\ \because \delta^7 f_{-1/2} &= \delta^6 f_0 - \delta^6 f_{-1} = f_3 - 6f_2 + 15f_1 - 20f_0 + 15f_{-1} - 6f_{-2} + f_{-3} - \\ &\quad - f_2 + 6f_1 - 15f_0 + 20f_{-1} - 15f_{-2} + 6f_{-3} - f_{-4} \\ &= f_3 - 7f_2 + 21f_1 - 35f_0 + 35f_{-1} - 21f_{-2} + 7f_{-3} - f_{-4}, \\ \therefore \delta^7 f_0 &= \frac{1}{2}[f_4 - 6f_3 + 14f_2 - 14f_1 + 14f_{-1} - 14f_{-2} + 6f_{-3} - f_{-4}]. \end{aligned} \tag{22}$$

$$\begin{aligned} \delta^9 f_0 &= \frac{1}{2}[\delta^9 f_{1/2} + \delta^9 f_{-1/2}], \\ \because \delta^9 f_{1/2} &= \delta^8 f_1 - \delta^8 f_0 = f_5 - 8f_4 + 28f_3 - 56f_2 + 70f_1 - 56f_0 + 28f_{-1} - 8f_{-2} + f_{-3} - \\ &\quad - f_4 + 8f_3 - 28f_2 + 56f_1 - 70f_0 + 56f_{-1} - 28f_{-2} + 8f_{-3} - f_{-4} \\ &= f_5 - 9f_4 + 36f_3 - 84f_2 + 126f_1 - 126f_0 + 84f_{-1} - 36f_{-2} + 9f_{-3} - f_{-4}, \\ \because \delta^9 f_{-1/2} &= \delta^8 f_0 - \delta^8 f_{-1} = f_4 - 8f_3 + 28f_2 - 56f_1 + 70f_0 - 56f_{-1} + 28f_{-2} - 8f_{-3} + f_{-4} - \\ &\quad - f_3 + 8f_2 - 28f_1 + 56f_0 - 70f_{-1} + 56f_{-2} - 28f_{-3} + 8f_{-4} - f_{-5} \\ &= f_4 - 9f_3 + 36f_2 - 84f_1 + 126f_0 - 126f_{-1} + 84f_{-2} - 36f_{-3} + 9f_{-4} - f_{-5}, \\ \therefore \delta^9 f_0 &= \frac{1}{2}[f_5 - 8f_4 + 27f_3 - 48f_2 + 42f_1 - 42f_{-1} + 48f_{-2} - 27f_{-3} + 8f_{-4} - f_{-5}]. \end{aligned} \tag{23}$$

Thus given values of a function f(x) at a finite number of discrete points n = 10, and letting p = [(x - x_i) / (x_{i+1} - x_i)], Stirling's central-difference interpolation formula can be expressed using functional values as

$$f_p = f_0 + p\delta f_0 + \frac{p^2}{2!}\delta^2 f_0 + \frac{p(p^2-1)}{3!}\delta^3 f_0 + \frac{p^2(p^2-1)}{4!}\delta^4 f_0 + \frac{p(p^2-1)(p^2-4)}{5!}\delta^5 f_0 + + \frac{p^2(p^2-1)(p^2-4)}{6!}\delta^6 f_0 + \frac{p(p^2-1)(p^2-4)(p^2-9)}{7!}\delta^7 f_0 + \frac{p^2(p^2-1)(p^2-4)(p^2-9)}{8!}\delta^8 f_0 + + \frac{p(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{9!}\delta^9 f_0 + \frac{p^2(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{10!}\delta^{10} f_0. \quad (24)$$

From above equations, we can write this equation as

$$\begin{aligned} f_p = {} & f_0 + \frac{p}{2}[f_1 - f_{-1}] + \frac{p^2}{2}[f_1 - 2f_0 + f_{-1}] + \frac{p(p^2-1)}{12}[f_2 - 2f_1 + 2f_{-1} - f_{-2}] + \\ & + \frac{p^2(p^2-1)}{24}[f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}] + \\ & + \frac{p(p^2-1)(p^2-4)}{240}[f_3 - 4f_2 + 5f_1 - 5f_{-1} + 4f_{-2} - f_{-3}] + \\ & + \frac{p^2(p^2-1)(p^2-4)}{720}[f_3 - 6f_2 + 15f_1 - 20f_0 + 15f_{-1} + -6f_{-2} + f_{-3}] \\ & + \frac{p(p^2-1)(p^2-4)(p^2-9)}{10080}[f_4 - 6f_3 + 14f_2 - 14f_1 + 14f_{-1} - 14f_{-2} + 6f_{-3} - f_{-4}] + \\ & + \frac{p^2(p^2-1)(p^2-4)(p^2-9)}{40320}[f_4 - 8f_3 + 28f_2 - 56f_1 + 70f_0 - 56f_{-1} + 28f_{-2} - 8f_{-3} + f_{-4}] + \\ & + \frac{p(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{725760}\begin{bmatrix} f_5 - 8f_4 + 27f_3 - 48f_2 + 42f_1 - 42f_{-1} + 48f_{-2} - 27f_{-3} + \\ + 8f_{-4} - f_{-5} \end{bmatrix} \\ & + \frac{p^2(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{3628800}\begin{bmatrix} f_5 - 10f_4 + 45f_3 - 120f_2 + 210f_1 - 252f_0 + 210f_{-1} - \\ -120f_{-2} + 45f_{-3} - 10f_{-4} + f_{-5} \end{bmatrix}. \end{aligned} \quad (25)$$

Differentiate this equation with respect to p and put p = 0, then

$$\left.\frac{df(x_p, y_0)}{dp}\right|_{(x_0,y_0)} = \left.f'_p\right|_{(x_0,y_0),p=0} \quad (26)$$

$$\begin{aligned} \therefore \left.f'_p\right|_{(x_0,y_0),p=0} = {} & \frac{1}{2}[f_1 - f_{-1}] - \frac{1}{12}[f_2 - 2f_1 + 2f_{-1} - f_{-2}] + \\ & + \frac{1}{60}[f_3 - 4f_2 + 5f_1 - 5f_{-1} + 4f_{-2} - f_{-3}] - \\ & - \frac{1}{280}[f_4 - 6f_3 + 14f_2 - 14f_1 + 14f_{-1} - 14f_{-2} + 6f_{-3} - f_{-4}] + \\ & + \frac{1}{1260}[f_5 - 8f_4 + 27f_3 - 48f_2 + 42f_1 - 42f_{-1} + 48f_{-2} - 27f_{-3} + + 8f_{-4} - f_{-5}] \end{aligned} \quad (27)$$

Then

$$\begin{aligned} \left.f'_p\right|_{p=0} = {} & 7.9365 x 10^{-4}(f_5 - f_{-5}) - 9.92063 x 10^{-3}(f_4 - f_{-4}) + 0.0553571(f_3 - f_{-3}) - \\ & - 0.238095(f_2 - f_{-2}) + 0.75(f_1 - f_{-1}) \end{aligned} \quad (28)$$

Since x = x_0 +ph = x_p.

Then $\frac{dx}{dp} = h, \frac{dp}{dx} = \frac{1}{h}$

We get $\frac{df(x\ , y_0)}{dx} = f_x = \frac{df(x\ , y_0)}{dp} X \frac{dp}{dx} = f'_p X \frac{dp}{dx}$ (29)

Let the interval length h = 0.001

Then we can get f_x at a point(x_0, y_0) for the surface z =f (x, y)

$$\boxed{f_x = 100\begin{bmatrix} 7.9365 x 10^{-4}(f_5 - f_{-5}) - 9.92063 x 10^{-3}(f_4 - f_{-4}) + 0.0553571(f_3 - f_{-3}) - \\ - 0.238095(f_2 - f_{-2}) + 0.75(f_1 - f_{-1}) \end{bmatrix}.} \quad (30)$$

Differentiate f_p twice with respect to p and put p = 0, then

$$\left.\frac{d^2 f(x_p, y_0)}{dp^2}\right|_{(x_0,y_0)} = \left. f''_p \right|_{(x_0,y_0),p=0} \tag{31}$$

$$\begin{aligned} \therefore \left. f''_p \right|_{(x_0,y_0),p=0} &= [f_1 - 2f_0 + f_{-1}] - \frac{1}{12}[f_2 - 4f_1 + 6f_0 - 4f_{-1} + f_{-2}] + \\ &+ \frac{1}{90}[f_3 - 6f_2 + 15f_1 - 20f_0 + 15f_{-1} + -6f_{-2} + f_{-3}] - \\ &- \frac{1}{560}[f_4 - 8f_3 + 28f_2 - 56f_1 + 70f_0 - 56f_{-1} + 28f_{-2} - 8f_{-3} + f_{-4}] + \\ &+ \frac{1}{3150}[f_5 - 10f_4 + 45f_3 - 120f_2 + 210f_1 - 252f_0 + 210f_{-1} - 120f_{-2} + 45f_{-3} - 10f_{-4} + f_{-5}] \end{aligned} \tag{32}$$

Then

$$\begin{aligned} \left. f''_p \right|_{p=0} &= 3.17\,x10^{-4}(f_5 - f_{-5}) - 4.96\,x10^{-3}(f_4 - f_{-4}) + 0.039683(f_3 - f_{-3}) - \\ &- 0.238095(f_2 - f_{-2}) + 1.666667(f_1 - f_{-1}) - 2.927222 f_0 . \end{aligned} \tag{33}$$

Then we can get f_{xx} at a point(x_0, y_0) for the surface z = f (x, y)

$$f_{xx} = (100)^2 \begin{bmatrix} 3.17\,x10^{-4}(f_5 - f_{-5}) - 4.96\,x10^{-3}(f_4 - f_{-4}) + 0.039683(f_3 - f_{-3}) - \\ - 0.238095(f_2 - f_{-2}) + 1.666667(f_1 - f_{-1}) - 2.927222 f_0 \end{bmatrix}. \tag{34}$$

We can also find f_y and f_{yy} at a point (x_0, y_0) for the surface z = f(x, y) by using the same method

$$\text{Since } \left.\frac{df(x_0, y_p)}{dp}\right|_{(x_0,y_0)} = \left. \tilde{f}'_p \right|_{(x_0,y_0),p=0} , \quad \left.\frac{d^2 f(x_0, y_p)}{dp^2}\right|_{(x_0,y_0)} = \left. \tilde{f}''_p \right|_{(x_0,y_0),p=0} \tag{35}$$

Then we get

$$f_y = 100 \begin{bmatrix} 7.9365\,x10^{-4}(\tilde{f}_5 - \tilde{f}_{-5}) - 9.92063\,x10^{-3}(\tilde{f}_4 - \tilde{f}_{-4}) + 0.0553571(\tilde{f}_3 - \tilde{f}_{-3}) - \\ - 0.238095(\tilde{f}_2 - \tilde{f}_{-2}) + 0.75(\tilde{f}_1 - \tilde{f}_{-1}) \end{bmatrix}. \tag{36}$$

and

$$f_{yy} = (100)^2 \begin{bmatrix} 3.17\,x10^{-4}(\tilde{f}_5 - \tilde{f}_{-5}) - 4.96\,x10^{-3}(\tilde{f}_4 - \tilde{f}_{-4}) + 0.039683(\tilde{f}_3 - \tilde{f}_{-3}) - \\ - 0.238095(\tilde{f}_2 - \tilde{f}_{-2}) + 1.666667(\tilde{f}_1 - \tilde{f}_{-1}) - 2.927222 \tilde{f}_0 \end{bmatrix}. \tag{37}$$

To calculate f_{xy} at a point(x_0, y_0) for the surface z =f (x, y) by Stirling's central-difference interpolation formula which expressed using functional values as

$$\begin{aligned} f_p = f(x_0 + ph, y) &= f(x_0, y) + p\delta f(x_0, y) + \frac{p^2}{2!}\delta^2 f(x_0, y) + \frac{p(p^2-1)}{3!}\delta^3 f(x_0, y) + \frac{p^2(p^2-1)}{4!}\delta^4 f(x_0, y) + \\ &+ \frac{p(p^2-1)(p^2-4)}{5!}\delta^5 f(x_0, y) + \frac{p^2(p^2-1)(p^2-4)}{6!}\delta^6 f(x_0, y) + \\ &+ \frac{p(p^2-1)(p^2-4)(p^2-9)}{7!}\delta^7 f(x_0, y) + \frac{p^2(p^2-1)(p^2-4)(p^2-9)}{8!}\delta^8 f(x_0, y) + \\ &+ \frac{p(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{9!}\delta^9 f(x_0, y) + \frac{p^2(p^2-1)(p^2-4)(p^2-9)(p^2-16)}{10!}\delta^{10} f(x_0, y). \end{aligned} \tag{38}$$

Differentiate this equation with respect to p, then

$$\left.\frac{df(x_p, y\ )}{dp}\right|_{(x_0,y)} = \left.f'_p\right|_{(x_0,y)} \tag{39}$$

$$\begin{aligned}\therefore f'_p = f'_p(x_0+ph,y) = {} & \delta f(x_0,y) + p\delta^2 f(x_0,y) + \frac{3p^2-1}{6}\delta^3 f(x_0,y) + \frac{4p^3-2p}{24}\delta^4 f(x_0,y) + \\ & + \frac{5p^4-15p^2+4}{120}\delta^5 f(x_0,y) + \frac{6p^5-20p^3+8p}{720}\delta^6 f(x_0,y) + \\ & + \frac{7p^6-70p^4+147p^2-36}{5040}\delta^7 f(x_0,y) + \frac{8p^7-84p^5+196p^3-72p}{40320}\delta^8 f(x_0,y) + \\ & + \frac{9p^8-210p^6+1365p^4-2460p^2+576}{362880}\delta^9 f(x_0,y) + \\ & + \frac{10p^9-240p^7+1638p^5-3280p^3+1152p}{3628800}\delta^{10} f(x_0,y).\end{aligned} \tag{40}$$

Let $y = y_0 + qh$, $f(x_0, y_0) = f_0$

Then

$$f'_p(x_0+ph, y_0+qh) = \delta\begin{bmatrix} \tilde{f}_0 + q\delta\tilde{f}_0 + \frac{q^2}{2}\delta^2\tilde{f}_0 + \frac{q^3-q}{6}\delta^3\tilde{f}_0 + \\ + \frac{q^4-q^2}{24}\delta^4\tilde{f}_0 + \frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0 + \\ + \frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0 + \\ + \frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0 + \\ + \frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0 + \\ + \frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0 + \\ + \frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0 \end{bmatrix} +$$

$$+ p\delta^2\begin{bmatrix} \tilde{f}_0 + q\delta\tilde{f}_0 + \frac{q^2}{2}\delta^2\tilde{f}_0 + \frac{q^3-q}{6}\delta^3\tilde{f}_0 + \\ + \frac{q^4-q^2}{24}\delta^4\tilde{f}_0 + \frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0 + \\ + \frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0 + \\ + \frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0 + \\ + \frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0 + \\ + \frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0 + \\ + \frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0 \end{bmatrix} +$$

$$+ \frac{3p^2-1}{6}\delta^3\begin{bmatrix} \tilde{f}_0 + q\delta\tilde{f}_0 + \frac{q^2}{2}\delta^2\tilde{f}_0 + \frac{q^3-q}{6}\delta^3\tilde{f}_0 + \\ + \frac{q^4-q^2}{24}\delta^4\tilde{f}_0 + \frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0 + \\ + \frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0 + \\ + \frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0 + \\ + \frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0 + \\ + \frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0 + \\ + \frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0 \end{bmatrix} +$$

$$+\frac{4p^3-2p}{24}\delta^4\left[\begin{array}{l}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{array}\right]+$$

$$+\frac{5p^4-15p^2+4}{120}\delta^5\left[\begin{array}{l}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{array}\right]+$$

$$+\frac{6p^5-20p^3+8p}{720}\delta^6\left[\begin{array}{l}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}(x_0,y_0)+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}(x_0,y_0)\end{array}\right]+$$

$$+\frac{7p^6-70p^4+147p^2-36}{5040}\delta^7\left[\begin{array}{l}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{array}\right]+$$

$$+\frac{8p^7-84p^5+196p^3-72p}{40320}\delta^8\begin{bmatrix}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{bmatrix}+$$

$$+\frac{9p^8-210p^6+1365p^4-2460p^2+576}{725760}\delta^9\begin{bmatrix}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{bmatrix}+.$$

$$+\frac{10p^9-240p^7+1638p^5-3280p^3+1152p}{3628800}\delta^{10}\begin{bmatrix}\tilde{f}_0+q\delta\tilde{f}_0+\frac{q^2}{2}\delta^2\tilde{f}_0+\frac{q^3-q}{6}\delta^3\tilde{f}_0+\\+\frac{q^4-q^2}{24}\delta^4\tilde{f}_0+\frac{q^5-5q^3+4q}{120}\delta^5\tilde{f}_0+\\+\frac{q^6-5q^4+4q^2}{720}\delta^6\tilde{f}_0+\\+\frac{q^7-14q^5+49q^3-36q}{5040}\delta^7\tilde{f}_0+\\+\frac{q^8-14q^6+49q^4-36q^2}{40320}\delta^8\tilde{f}_0+\\+\frac{q^9-30q^7+273q^5-820q^3+576q}{362880}\delta^9\tilde{f}_0+\\+\frac{q^{10}-30q^8+273q^6-820q^4+576q^2}{3628800}\delta^{10}\tilde{f}_0\end{bmatrix}. \quad (41)$$

Differentiate f_p twice with respect to p, q and put p = 0, q = 0 then

$$\begin{aligned}f''_p\big|_{(x_0,y_0),p=0,q=0} &= \delta\left[\delta\tilde{f}_0-\frac{1}{6}\delta^3\tilde{f}_0+\frac{1}{30}\delta^5\tilde{f}_0-\frac{1}{140}\delta^7\tilde{f}_0+\frac{1}{630}\delta^9\tilde{f}_0\right]-\\&-\frac{1}{6}\delta^3\left[\delta\tilde{f}_0-\frac{1}{6}\delta^3\tilde{f}_0+\frac{1}{30}\delta^5\tilde{f}_0-\frac{1}{140}\delta^7\tilde{f}_0+\frac{1}{630}\delta^9\tilde{f}_0\right]+\\&+\frac{1}{30}\delta^5\left[\delta\tilde{f}_0-\frac{1}{6}\delta^3\tilde{f}_0+\frac{1}{30}\delta^5\tilde{f}_0-\frac{1}{140}\delta^7\tilde{f}_0+\frac{1}{630}\delta^9\tilde{f}_0\right]-\\&-\frac{1}{140}\delta^7\left[\delta\tilde{f}_0-\frac{1}{6}\delta^3\tilde{f}_0+\frac{1}{30}\delta^5\tilde{f}_0-\frac{1}{140}\delta^7\tilde{f}_0+\frac{1}{630}\delta^9\tilde{f}_0\right]+\\&+\frac{1}{630}\delta^9\left[\delta\tilde{f}_0-\frac{1}{6}\delta^3\tilde{f}_0+\frac{1}{30}\delta^5\tilde{f}_0-\frac{1}{140}\delta^7\tilde{f}_0+\frac{1}{630}\delta^9\tilde{f}_0\right].\end{aligned} \quad (42)$$

We get:

$$\begin{aligned}f''_p\big|_{(x_0,y_0),p=0,q=0} &= \delta^2\tilde{f}_0+\delta^4\tilde{f}_0\left(-\frac{1}{6}-\frac{1}{6}\right)+\delta^6\tilde{f}_0\left(\frac{1}{30}+\frac{1}{36}+\frac{1}{30}\right)+\\&+\delta^8\tilde{f}_0\left(-\frac{1}{140}-\frac{1}{180}-\frac{1}{140}-\frac{1}{180}\right)+\\&+\delta^{10}\tilde{f}_0\left(\frac{1}{630}+\frac{1}{840}+\frac{1}{900}+\frac{1}{840}+\frac{1}{630}\right).\end{aligned} \quad (43)$$

Then,

$$
\begin{aligned}
f''_p\Big|_{(x_0,y_0),p=0,q=0} &= \tilde{f}_1 - 2\tilde{f}_0 + \tilde{f}_{-1} - 0.33333\left[\tilde{f}_2 - 4\tilde{f}_1 + 6\tilde{f}_0 - 4\tilde{f}_{-1} + \tilde{f}_{-2}\right] + \\
&+ 0.09444\left[\tilde{f}_3 - 6\tilde{f}_2 + 15\tilde{f}_1 - 20\tilde{f}_0 + 15\tilde{f}_{-1} - 6\tilde{f}_{-2} + \tilde{f}_{-3}\right] - \\
&- 0.0254\left[\tilde{f}_4 - 8\tilde{f}_3 + 28\tilde{f}_2 - 56\tilde{f}_1 + 70\tilde{f}_0 - 56\tilde{f}_{-1} + 28\tilde{f}_{-2} - 8\tilde{f}_{-3} + \tilde{f}_{-4}\right] + \\
&+ 6.66667x10^{-3}\begin{bmatrix} \tilde{f}_5 - 10\tilde{f}_4 + 45\tilde{f}_3 - 120\tilde{f}_2 + 210\tilde{f}_1 - 252\tilde{f}_0 + 210\tilde{f}_{-1} - \\ -120\tilde{f}_{-2} + 45\tilde{f}_{-3} - 10\tilde{f}_{-4} + \tilde{f}_{-5}. \end{bmatrix}.
\end{aligned} \tag{44}
$$

Hence,

$$
\begin{aligned}
f''_p\Big|_{(x_0,y_0),p=0,q=0} &= -7.83562\tilde{f}_0 + 5.31302(\tilde{f}_1 + \tilde{f}_{-1}) - 1.69157\left(\tilde{f}_2 + \tilde{f}_{-2}\right) + \\
&+ 0.32779\left[\tilde{f}_3 + \tilde{f}_{-3}\right] - 0.0321\left[\tilde{f}_4 + \tilde{f}_{-4}\right] + 6.66667x10^{-3}\left[\tilde{f}_5 + \tilde{f}_{-5}.\right].
\end{aligned} \tag{45}
$$

Let h = 0.001

Then we can get f_{xy} at a point(x_0, y_0) for the surface z =f (x, y)

$$
f_{xy} = (100)^2\begin{bmatrix} -7.83562\tilde{f}_0 + 5.31302(\tilde{f}_1 + \tilde{f}_{-1}) - 1.69157\left(\tilde{f}_2 + \tilde{f}_{-2}\right) + \\ +0.32779\left[\tilde{f}_3 + \tilde{f}_{-3}\right] - 0.0321\left[\tilde{f}_4 + \tilde{f}_{-4}\right] + 6.66667x10^{-3}\left[\tilde{f}_5 + \tilde{f}_{-5}.\right]. \end{bmatrix} \tag{46}
$$

V. Conclusions

The most important central difference formula is stirling interpolation and it takes the mean of Gauss forward and backward formulae. Stirling 's formula gives the most accurate result for small p and interpolation near the middle of a table.

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