IOSR Journal of Mathematics (IOSR-JM)
ISSN: 2278-5728. Volume 4, Issue 1 (Nov. - Dec. 2012), PP 07-12
www.iosrjournals.org

Hamiltonian Laceability in a Class of 4-Regular Graphs

Girisha .A[1], H. Mariswamy[2], R. Murali[3], G. Rajendra[4]
[1] *Department of Mathematics, Acharya Institute of Technology, Karnataka, India,*
[2] *Department of Mathematics, BBMP Junior College, Karnataka, India.*
[3] *Department of Mathematics, Dr. Ambedkar Institute of Technology, Karnataka, India .*
[4] *Department of Industrial Engineering and Management, Dr. Ambedkar Institute of Technology, Karnataka, India.*

Abstract : *B. Alspach, C.C. Chen and Kevin Mc Avaney [1] have discussed the Hamiltonian laceability of the Brick product C(2n, m, r) for even cycles. In [2], the authors have shown that the (m,r)-Brick Product C(2n+1, 1, 2) is Hamiltonian-t-laceable for* $1 \leq t \leq diamC_{2n+1}$. *In this paper we explore the Hamiltonian-t-laceability of the (m,r)-Brick Product C(2n+1,1,r) for r=3 and 4.*
Keywords - *Brick product, Connected graph, Hamiltonian-t-laceable graph.*
2000 Mathematics subject classification: *05C45, 05C99*

I. Introduction

Let G be a finite, simple, connected and undirected graph. Let u and v be two vertices in G. The distance between u and v denoted by d(u,v) is the length of a shortest u-v path in G. G is **Hamiltonian laceable** if there exists a Hamiltonian path between every pair of vertices in G at an odd distance. G is **Hamiltonian-t-laceable** if there exists a Hamiltonian path between every pair of vertices u and v in G with d(u,v)=t , $1 \leq t \leq$ diamG. In [1], B. Alspach, C.C. Chen and Kevin McAvaney have explored Hamiltonian Laceability in the Brick Products of even cycles. In [2], Leena Shenoy and R. Murali have discussed the (m,r)-Brick Product of odd cycles C(2n+1,m,r). In this paper we explore the Hamiltonian-t-laceability of the (m,r)-Brick Product C(2n+1,1,r) for r=3 and 4.

Definition 1: Let m, n and r be a positive integers. Let C_{2n}- $a_0,a_1,a_2,a_3 \ldots a_{(2n-1)}a_0$ denote a cycle of order 2n. The (m,r)-brick product of C_{2n} denoted by C(2n,m,r) is defined for m=1, we require that r be odd and greater than 1. Then C(2n,m,r) is obtained from C_{2n} by adding chords $a_{2k}(a_{2k+r})$, k=1,2,….n, where the computation is performed under modulo 2n.

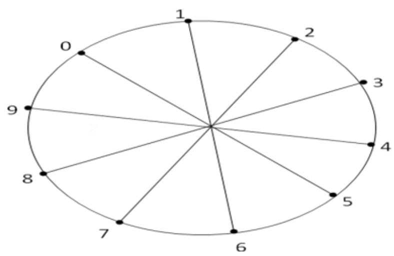

Fig. 1: Brick product C(10,1,5)

Definition 2: Let m,n and r be positive integers. Let $C_{2n+1}= a_0\, a_1\, a_2\, a_3 \ldots a_{2n}\, a_0$ denote a cycle of order 2n+1 (n>1). The (m,r)-brick product of C_{2n+1}, denoted by C(2n+1,m,r) is defined for m=1, we require that 1< r < 2n. Then C(2n+1,m,r) is obtained from C_{2n+1} by adding chords $a_k(a_{k+r})$, $0 \leq k \leq 2n$ where the computation is performed under modulo 2n+1.

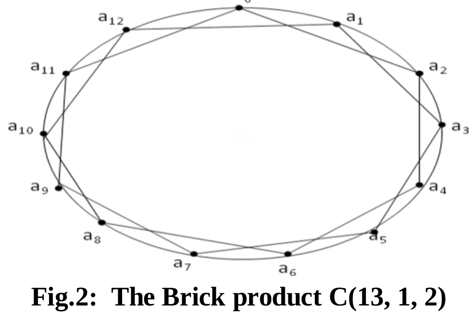

Fig.2: The Brick product C(13, 1, 2)

Definition 3: Let u and v be two distinct vertices in a connected graph G. Then u and v are attainable in G if there exists a Hamiltonian path in G from u and v.

Terminologies: For m=1, if a_i is any vertex of C(2n+1,m,r), then the following are defined.

$(a_i)\ P[m] = (a_i)(a_{i+1})(a_{i+2}) \ldots\ldots.(a_{i+m-1})$ $\forall\, i \in \mathbb{Z}$

$(a_i)\ P^{-1}[m] = (a_i)(a_{i-1})(a_{i-2}) \ldots\ldots (a_{i-m+1})$ $\forall\, i \in \mathbb{Z}$

$(a_i)\ [J] = (a_i)(a_{i+r})$ and $(a_i)[J^{-1}] = (a_i)(a_{i-r})$ $\forall\, i \in \mathbb{Z}$

Example: For n= 4, C(2n+1, 1, 4) and $d(a_i, a_j) = 2$ for i =1 and j =3, the Hamiltonian path is given by $(a_i)\ P(2)\ J\ [P^{-1}(2)]^2\ J^{-1}\ [P^{-1}(2)]^{2(n-3)}\ J^{-1} = a_1\text{-}a_2\text{-}a_6\text{-}a_5\text{-}a_4\text{-}a_0\text{-}a_8\text{-}a_7\text{-}a_3$ under modulo 2n+1.

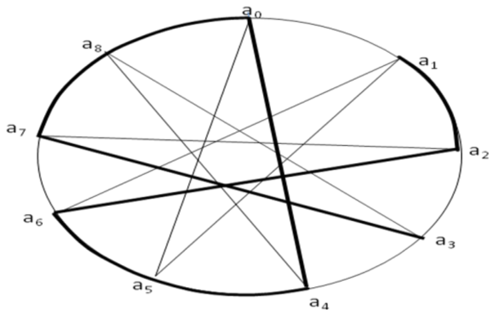

Fig.3: Hamiltonian path from vertex a_1 to a_3 in the Brick Product C(9,1,4)

In [2], Leena Shenoy and R. Murali proved the following theorem.

Theorem 1: C(2n+1, 1, 2) is Hamiltonian – t – laceable. Where $1 \le t \le$ diam G.
We now prove the following results.

II. Results

Theorem2: The graph C(2n+1, 1, 3) is Hamiltonian-t-laceable for t=1,2 if n=3 and is Hamiltonian-t-laceable for t=1,2,3 if n≥6 such that $(2n+1) \equiv 1 \pmod 3$.

Proof: Consider the graph G= C(2n+1, 1, 3).
Let $d(a_i, a_j) = t$, $(0 \le i < j \le 2n)$. For convenience we take j>i. Here we need to establish the following claims to show that a_i and a_j are attainable for t=1, 2 and 3.

<u>Claim 1:</u> t =1

Case i: j - i = 1 or (2n+1)-(j-i) =1
If j - i = 1 in C_{2n+1} then, a_i and a_j are attainable in G, since $(a_i)\ [P^{-1}(2)]^{2n}$ is the Hamiltonian path.
If (2n+1)-(j - i) =1 in C_{2n+1} then, a_i and a_j are attainable in G, since $(a_i)\ [P(2)]^{2n}$ is the Hamiltonian path.

Case(ii): j - i = 3 or (2n+1)-(j - i) =3
If j - i = 3 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [P^{-1}(2)]^{2n-3}\ J^{-1}(P)^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =3 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [P(2)]^{2n-3}\ J(P^{-1})^2$ is the Hamiltonian path.

<u>Claim 2:</u> t = 2

Case(i): j - i = 2 or (2n+1)-(j - i) =2
If j - i = 2 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [P^{-1}(2)\ J^{-1}\ P(2)\ J^{-1})^{n/3}\ [J^{-1}]^{2n/3}$ is a Hamiltonian path.
If (2n+1)-(j - i) =2 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [P(2)\ J\ P^{-1}(2)\ J)^{n/3}\ [J]^{2n/3}$ is a Hamiltonian path.

Case(ii): j - i = 4 or (2n+1)-(j - i) = 4
If j - i = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [J^{-1}]^{2(n-3)/3}\ [P^{-1}(2)\ J^{-1}\ P^{-1}(2)P^{-1}(2)]\ [J^{-1}\ P(2)\ J^{-1}\ P^{-1}(2)]^{n-3/3}\ J^{-1}\ P^{-1}(2)$ is the Hamiltonian path.
If (2n+1)-(j - i) = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [J]^{2(n-3)/3}\ [P(2)\ J\ P(2)P(2)]\ [J\ P^{-1}(2)\ J\ P(2)]^{n-3/3}\ J\ P(2)$ is the Hamiltonian path.

Case(iii): If j - i = 6 or (2n+1)-(j - i) =6
If j - i = 6 in C_{2n+1} then, a_i and a_j are attainable in G, since
$(a_i)\ [P(2)\ J^{-1}\ P^{-1}(2)\ J^{-1}]^{n-3/3}\ [P(2)\ J^{-1}\ [P^{-1}(2)]^3]\ [J^{-1}]^{2n-3/3}$ is the Hamiltonian path.

If $(2n+1)-(j-i)=6$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)\ J\ P(2)\ J]^{n-3/3}\ [P^{-1}(2)\ J\ [P(2)]^3]\ [J]^{2n-3/3}$ is the Hamiltonian path.
Claim 3: t=3
Case(i): $j-i=5$ or $(2n+1)-(j-i)=5$
If $j-i=5$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)\ J\ P^{-1}(2)\ J]\ [P(2)]^{2(n-3)}\ [J]^2$ is the Hamiltonian path.
If $(2n+1)-(j-i)=5$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)\ J^{-1}\ P(2)\ J^{-1}]\ [P^{-1}(2)]^{2(n-3)}\ [J^{-1}]^2$ is the Hamiltonian path.
Case(ii): $j-i=7$ or $(2n+1)-(j-i)=7$
If $j-i=7$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-7}\ [J^{-1}\ P^{-1}(2)]^2\ [P(2)\ J\ P(2)]$ is the Hamiltonian path.
If $(2n+1)-(j-i)=7$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-7}\ [J^{-1}\ P^{-1}(2)]^2\ [P(2)\ J\ P(2)]$ is the Hamiltonian path.
Case(iii): $j-i=9$ or $(2n+1)-(j-i)=9$
If $j-i=9$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)\ J^{-1}]\ [P^{-1}(2)]^{2n-11}\ [J^{-1}\ P(2)J^{-1}\ P^{-1}(2)\ J^{-1}\ [P(2)]^2\ [J]^2]$ is the Hamiltonian path.
If $(2n+1)-(j-i)=9$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)\ J]\ [P(2)]^{2n-11}\ [J\ P^{-1}(2)J\ P(2)\ J\ [P^{-1}(2)]^2\ [J^{-1}]^2]$ is the Hamiltonian path.

Hence the proof . ■

Theorem3: The graph C(2n+1, 1, 3) is Hamiltonian-t-laceable for t=1,2,3. Where n≥5 such that (2n+1) ≡ 2 (mod 3).

Proof: Consider a graph G= C(2n+1, 1, 3).
Let $d(a_i, a_j)=t$, Here we need to establish the following claims to show that a_i and a_j $(0 \le i < j \le 2n)$ are attainable for t=1,2,3.
Claim 1: t=1
Case(i): $j-i=1$ or $(2n+1)-(j-i)=1$
If $j-i=1$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n}$ is the Hamiltonian path.
If $(2n+1)-(j-i)=1$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n}$ is the Hamiltonian path.
Case(ii): If $j-i=3$ or $(2n+1)-(j-i)=3$
If $j-i=3$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1}]^{2n}$ is the Hamiltonian path.
If $(2n+1)-(j-i)=3$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J]^{2n}$ is the Hamiltonian path.
Claim 2: t=2
Case(i): If $j-i=2$ or $(2n+1)-(j-i)=2$
If $j-i=2$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)\ J^{-1}\ P(2)\ J^{-1})\ P^{-1}(2)\ [J^{-1}]^{2(n+1)/3}\ P^{-1}(2)$ is the Hamiltonian path.
If $(2n+1)-(j-i)=2$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)\ J\ P^{-1}(2)\ J)\ P(2)\ [J]^{2(n+1)/3}\ P(2)$ is the Hamiltonian path.
Case(ii): $j-i=4$ or $(2n+1)-(j-i)=4$
If $j-i=4$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $P(2)\ [J^{-1}\ P^{-1}(2)\ J^{-1}\ P(2)]^{n-2/3}\ [J^{-1}\ P^{-1}(2)][J^{-1}]^{2n-1/3}$ is the Hamiltonian path.
If $(2n+1)-(j-i)=4$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $P^{-1}(2)\ [J\ P(2)\ J\ P^{-1}(2)]^{n-2/3}\ [J\ P(2)][J]^{2n-1/3}$ is the Hamiltonian path.
Case(iii): $j-i=6$ or $(2n+1)-(j-i)=6$
If $j-i=6$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)\ J^{-1}\ P(2)J^{-1}]^{n-2/3}\ [P^{-1}(2)]^4\ [J^{-1}]^{2(n-2)/3}$ is the Hamiltonian path.
If $(2n+1)-(j-i)=6$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)\ J\ P^{-1}(2)J]^{n-2/3}\ [P(2)]^4\ [J]^{2(n-2)/3}$ is the Hamiltonian path.
Claim 3: t=3
Case(i): $j-i=5$ or $(2n+1)-(j-i)=5$
If $j-i=5$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-5}\ [J^{-1}\ P(2)J^{-1}\ P(2)\ J]$ is the Hamiltonian path.
If $(2n+1)-(j-i)=5$ in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-5}\ [J\ P^{-1}(2)J\ P^{-1}(2)\ J^{-1}]$ is the Hamiltonian path.

Case(ii): j - i = 7 or (2n+1)-(j - i)=7
If j - i = 7 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-7}$ $[J^{-1} P(2)J^{-1} [P^{-1}(2)]^2 [J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i)=7 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-7}$ $[J P^{-1}(2)J [P(2)]^2 [J^{-1}]^2$ is the Hamiltonian path.
Case(iii): j - i = 9 or (2n+1)-(j - i) =9
If j - i = 9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1}]^{2n-9}$ $[J^{-1} P(2) J^{-1} P^{-1}(2) J^{-1} [P(2)]^2 [J]^2]$ is the Hamiltonian path.
If (2n+1)-(j - i) =9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J]^{2n-9}$ $[J P^{-1}(2) J P(2) J [P^{-1}(2)]^2 [J^{-1}]^2]$ is the Hamiltonian path.

Hence the proof ■

Theorem4: The graph C(2n+1, 1, 4) is Hamiltonian-t-laceable for t=1,2 if n = 4 and is Hamiltonian-t-laceable for t=1,2,3 if n ≥ 6 such that (2n+1) ≡ 1 (mod 4).

Proof: Consider a graph G= C(2n+1, 1, 4).
Let d(i, j)=t. Here we need to establish the following claims to show that a_i and a_j ($0 \le i < j \le 2n$)are attainable for t=1,2 and 3.

Claim1: t=1
Case(i): j - i = 1 or (2n+1)-(j - i) = 1
If j - i = 1 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n}$ is the Hamiltonian path.
If (2n+1)-(j - i) = 1 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n}$ is the Hamiltonian path.
Case(ii): j - i = 4 or (2n+1)-(j - i) = 4
If j - i = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1}]^{2n}$ is the Hamiltonian path.
If (2n+1)-(j - i) = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J]^{2n}$ is the Hamiltonian path.

Claim 2: t=2
Case(i): j - i = 2 or (2n+1)-(j - i) =2
If j - i = 2 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J]^n P(2) [J]^{n-1/2} P^{-1}(2) [J^{-1}]^{n-3/2}$ is the Hamiltonian path.
If (2n+1)-(j - i) =2 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1}]^n P^{-1}(2) [J^{-1}]^{n-1/2} P(2) [J]^{n-3/2}$ is the Hamiltonian path.
Case(ii): If j - i = 3 or (2n+1)-(j - i) =3
If j - i = 3 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J P(2) J^{-1} P(2) J] [P(2)]^{2(n-3)} J$ is the Hamiltonian path.
If (2n+1)-(j - i) =3 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1} P^{-1}(2) J P^{-1}(2) J^{-1}] [P^{-1}(2)]^{2(n-3)} J^{-1}$ is the Hamiltonian path.
Case(iii): j - i = 5 or (2n+1)-(j - i) = 5
If j - i = 5 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^2 J [P(2)]^{2(n-3)} J [P(2)]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) = 5 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^2 J^{-1} [P^{-1}(2)]^{2(n-3)} J^{-1} [P^{-1}(2)]^2$ is the Hamiltonian path.
Case(iv): j - i = 8 or (2n+1)-(j - i) = 8
If j - i = 8 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^2 J [P^{-1}(2)]^2 J [P(2)]^{2(n-4)} [J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) = 8 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^2 J^{-1} [P(2)]^2 J^{-1} [P^{-1}(2)]^{2(n-4)} [J^{-1}]^2$ is the Hamiltonian path.

Claim 3: t=3
Case(i): j - i = 6 or (2n+1)-(j - i) = 6
If j - i = 6 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-3}$ $[J^{-1} [P(2)]^2 J^{-1}P(2) J]$ is the Hamiltonian path.
If (2n+1)-(j - i) = 6 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-3}$ $[J [P^{-1}(2)]^2 J P^{-1}(2) J^{-1}]$ is the Hamiltonian path.
Case(ii): j - i = 7 or (2n+1)-(j - i) =7
If j - i = 7 in C_{2n+1}then, a_i and a_j are attainable in G, since

(a_i) $[P(2)]^2$ J $[P^{-1}(2)]^2$ J $[P(2)]^{2(n-4)}$ $[J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =7 in C_{2n+1}then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^2$ J^{-1} $[P(2)]^2$ J^{-1} $[P^{-1}(2)]^{2(n-4)}$ $[J^{-1}]^2$ is the Hamiltonian path.
Case(iii): j - i = 9 or (2n+1)-(j - i) =9
If j - i = 9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [P^{-1} (2)]$^{2n-9}$ J $[P(2)]^2$ J^{-1} $[P^{-1}(2)]^3$ $[J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [P(2)]$^{2n-9}$ $J^{-1}$$[P^{-1}(2)]^2$ J $[P(2)]^3$ $[J^{-1}]^2$ is the Hamiltonian path.

Case(iv): j - i = 12 or (2n+1)-(j - i)=12
If j - i = 12 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [P^{-1} (2)]$^{2(n-6)}$ J^{-1} $[P(2)]^2$ J^{-1} $[P^{-1}(2)]^2$ $J^{-1}$$[P(2)]^3$ $[J^{-1}]^2$ is the Hamiltonian path.
If (2n+1)-(j - i)=12 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [P(2)]$^{2(n-6)}$ J $[P^{-1}(2)]^2$ J $[P(2)]^2$ J$[P^{-1}(2)]^3$ $[J]^2$ is the Hamiltonian path.

Hence the proof . ■

Theorem5: The graph C(2n+1, 1, 4) is Hamiltonian-t-laceable for t=1,2,3. Where n ≥ 5 such that (2n+1) ≡ 3 (mod 4).

Proof: Consider a graph G= C(2n+1, 1, 4).
Let $d(a_i, a_j)$ = t. Here we need to establish the following claims to show that a_i and a_j $(0 \le i < j \le 2n)$ are attainable for t=1,2,3.

Claim 1: t=1
Case(i): j - i = 1 or (2n+1)-(j - i) = 1
If j - i = 1 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n}$ is the Hamiltonian path.
If (2n+1)-(j - i) = 1 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n}$ is the Hamiltonian path.
Case(ii): j - i = 4 or (2n+1)-(j - i) = 4
If j - i = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J^{-1}]^{2n}$ is the Hamiltonian path.
If (2n+1)-(j - i) = 4 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[J]^{2n}$ is the Hamiltonian path.

Claim 2: t=2
Case(i): j - i = 2 or (2n+1)-(j - i) =2
If j - i = 2 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) P(2) J $[P^{-1}(2)]^2$ J^{-1} $[P^{-1}(2)]^{2(n-3)}$ J^{-1} is the Hamiltonian path.
If (2n+1)-(j - i) =2 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $P^{-1}(2)$ J^{-1} $[P(2)]^2$ J $[P(2)]^{2(n-3)}$ J is the Hamiltonian path.
Case(ii): j - i = 3 or (2n+1)-(j - i) =3
If j - i = 3 in C_{2n+1}then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-5}$ J^{-1} $P^{-1}(2)$ J [$P^{-1}(2)]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =3 in C_{2n+1}then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-5}$ J P(2) J^{-1} [P(2)$]^2$ is the Hamiltonian path.
Case(iii): j - i = 5 or (2n+1)-(j - i) =5
If j - i = 5 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [J $P^{-1}(2)$ J^{-1}] $[P^{-1}(2)]^{2(n-3)}$ [J^{-1} $P^{-1}(2)$ J] is the Hamiltonian path.
If (2n+1)-(j - i) =5 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) [J^{-1} P(2) J] $[P(2)]^{2(n-3)}$ [J P(2) J^{-1}] is the Hamiltonian path.
Case(iv): j - i = 8 or (2n+1)-(j - i)= 8
If j - i = 8 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^2$ J P(2) $[J^{-1}]^2$$[P^{-1}(2)]^{2n-9}$ J^{-1} $P^{-1}(2)$ J is the Hamiltonian path.
If (2n+1)-(j - i)= 8 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^2$ J^{-1} $P^{-1}(2)$ $[J^{-1}]^2$$[P(2)]^{2n-9}$ J P(2) J^{-1} is the Hamiltonian path.

Claim 3: t=3
Case(i): j - i = 6 or (2n+1)-(j - i) =6
If j - i = 6 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-3}$ [J^{-1} $[P(2)]^2$ J^{-1}P(2) J is the Hamiltonian path.

If (2n+1)-(j - i) =6 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-3}$ $[J\, [P^{-1}(2)]^2\, J\, P^{-1}(2)\, J^{-1}$ is the Hamiltonian path.
Case(ii): j - i = 7 or (2n+1)-(j - i) =7
If j - i = 7 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P^{-1}(2)]^{2n-7}\, J^{-1}\, [P^{-1}(2)]^3\, J\, [P(2)]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =7 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[P(2)]^{2n-7}\, J\, [P(2)]^3\, J^{-1}\, [P^{-1}(2)]^2$ is the Hamiltonian path.
Case(iii): j - i = 9 or (2n+1)-(j - i) =9
If j - i = 9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[\, P^{-1}(2)]^{2n-9}\, J^{-1}\, [P(2)]^2\, J^{-1}\, [P^{-1}(2)]^3\, [J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i) =9 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[\, P(2)]^{2n-9}\, J\, [P^{-1}(2)]^2\, J\, [P(2)]^3\, [J^{-1}]^2$ is the Hamiltonian path.
Case(iv): j - i = 12 or (2n+1)-(j - i)=12
If j - i = 12 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[\, P^{-1}(2)]^{2(n-6)}\, J^{-1}\, [P(2)]^2\, J^{-1}\, [P^{-1}(2)]^2\, J^{-1}[P(2)]^3\, [J]^2$ is the Hamiltonian path.
If (2n+1)-(j - i)=12 in C_{2n+1} then, a_i and a_j are attainable in G, since
(a_i) $[\, P(2)]^{2(n-6)}\, J\, [P^{-1}(2)]^2\, J\, [P(2)]^2\, J\, [P^{-1}(2)]^3\, [J^{-1}]^2$ is the Hamiltonian path.

Hence the proof. ■

III. Conclusion

In this paper, we have proved that the (m,r)-Brick Product C(2n+1, 1, r) for r = 3, 4 is Hamiltonian-t-laceable for t= 1,2,3. The general problem whether C(2n+1, 1, r) for $1 < r \leq diam C_{2n+1}$ is Hamiltonian-t-laceable still remains open.

IV. Acknowledgements

The first author thankfully acknowledges the support and encouragement provided by the Management and the staff of the department of Mathematics, Acharya Institute of Technology, Bangalore. The authors are also thankful to the management and R&D centre, Department of Mathematics, Dr. Ambedkar Institute of Technology, Bangalore.

References

[1] Brain Alspach et al., On a class of Hamiltonian laceable 3-regular graphs, *Journal of Discrete Mathematics* 151(1996), 19-38.
[2] Leena N. shenoy and R.Murali, Laceability on a class of Regular Graphs, *International Journal of computational Science and Mathematics, volume 2, Number* 3 (2010), 397-406.