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Topic: Ties Adjusted Extended Sign Test For Ordered Data

[1]Oyeka, I. C.A And [2]Umeh, E.U
[1,2]Department Of Statistics, Nnamdi Azikiwe University, Awka

Abstract: *This paper developed a Ties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition that takes account of all possible pairwise combinations of treatment levels. A test statistic is developed to determine whether subjects are increasingly performing better or worse over time or space. The proposed method also enables the researcher to have a bird's eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of any desired interventionist measures. The method is illustrated with some data and shown to be more powerful than Friedman test and shown to beeasier to use than the Bartholomew procedure.*

I. Introduction:

If one has repeated measures randomly drawn from a number of related populations that are dependent on some demographic factors or conditions or that are ordered in time or space which do not satisfy the necessary assumptions for the use of parametric tests, then use of nonparametric methods is indicated and preferable. These types of data include subjects' or candidate's scores in examinations or job placement interviews at various points in time; diagnostic test results repeated a certain number of times; commodity prizes at various times, locations or markets; etc.

Statistical analysis of these types of data often require the use of non parametric methods such as Friedman's two way analysis of variance test by ranks or the Cochran's Q test(Gibbons 1971,Gibbons 1993, Oyeka et al, 2010; Siegel, 1956; Hollander and Wolfe, 1999, Freidlin and Gastwirth 2000).

However, a problem with these two methods is that the Friedman's test often tries to adjust for ties that occur in blocks or batches of sample observations by assigning these tied observations their mean ranks a procedure that tends to reduce the power of the test, while the Cochran's Q test requires the observations to be dichotomous assuming only two possible values. Furthermore, if the null hypothesis to be tested is that subjects are increasingly performing better or worse with time or space, then these two statistical procedures may not be readily applicable.

In this case the methods developed by Bartholomew and others (Bartholomew 1959a, 1959b) may then be used. However, some of these methods are rather difficult to apply in practice and the resolution of any ties that may occur within blocks of observations is not often easy.

In this paper, we propose aties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition, that takes account of all possible pairwise combinations of treatment levels.

II. The Proposed Method

Suppose $(x_{i1}, x_{i2}, \dots, x_{in})$ is the i^{th} batch, set of or match observations randomly drawn from related populations, $X_{1,}, X_2, \dots, X_k$ for $i = 1, 2, \dots, n$ where 'k' may be indexed in time or space. Populations $X_{1,}, X_2, \dots, X_k$ may be measurements on as low as the ordinal scale and need not be continuous.

The problem of research interest here is to determine whether subjects are on the average progressively increasing, experiencing no change or worsening in their scores or performance over time, space or remission of condition. It is quite possible that within any specified time interval say some subject scores at some time in the intervalinstead of as expected being monotone, other increasing or decreasing , may be higher than their scores earlier in the interval which are themselves higher than their scores later in the interval.

Let

$$d_{ij} = c \qquad \dots\dots 1$$

For $i = 1, 2, \dots, n;\ j = 1, 2, \dots, k-1$

Let

$$u_{ij} = \begin{cases} 1, if\ d_{ij} > 0 \\ 0, if\ d_{ij} = 0 \\ -1\ if\ d_{ij} < 0 \end{cases} \qquad \dots\dots 2$$

For $i = 1, 2, \dots, n;\ j = 1, 2, \dots, k-1$

Note that equations 1 and 2 can be combined into one equation as

$$u_{ij} = \begin{cases} 1, if\ x_{ij} > x_{ij+1} \\ 0,\ if\ x_{ij} = x_{ij+1} \\ -1\ if\ x_{ij} < x_{ij+1} \end{cases} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.. \text{2b}$$

For $i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, k-1$ which is easier to use when the data being analysed are ordinal scale data that are non- numeric measurements such as letter grades.

Define $\pi_j^+ = P(u_{ij} = 1);\ \pi_j^0 = P(u_{ij} = 0);\ \pi_j^- = P(u_{ij} = -1)$ …………………………… 3

Where

$\pi_j^+ + \pi_j^0 + \pi_j^- = 1$ ……………………………… 4

For $j = 1, 2, \ldots, k-1$

Note that equation2, 4 have structurally and intrinsically provided for the possibility of tied observations between successive pairs of the sampled populations. The model specifications allow ties to occur between these pairs of sampled populations with probability $\pi_j^0,\ j = 1, 2, \ldots, k-1$

Let

$W_j = \sum_{i=1}^{n} u_{ij}$ …………………………….. 5

Also let

$W = \sum_{i=1}^{k-1} w_j = \sum_{j=1}^{k-1} \sum_{i=1}^{n}$ ……………………….6

Now

$E(u_{ij}) = \pi_j^+ - \pi_j^-; Var(u_{ij}) = \pi_j^+ + \pi_j^- - (\pi_j^+ - \pi_j^-)^2$ ……………………. 7

Note that $\pi_j^+,\ \pi_j^- and\ \pi_j^-$ are respectively the probabilities that observations from greater than, equal to, less than observations from population X_{j+1} for $j = 1, 2, \ldots, k-1$. The sample estimates of these probabilities are$\hat{\pi}_j^+ = \frac{f_j^+}{n};\ \hat{\pi}_j^0 = \frac{f_j^0}{n};\ \hat{\pi}_j^- = \frac{f_j^-}{n}$ …………………………… 8

Where $f_j^+, f_j^0 and\ f_j^-$ are respectively the number of the $1s, 0s\ and - 1s$ in the frequency distribution of these numbers in u_{ij} for $i = 1, 2, \ldots, n, j = 1, 2, \ldots, k-1$

From equation 5 we have that the expected value of $W_j\ is$

$E(w_j) = \sum_{i=1}^{n} E(u_{ij}) = n(\pi_j^+ - \pi_j^-)$ …………………….. 9

And

$Var(w_j) = \sum_{i=1}^{n} Var(u_{ij}) = n\left(\pi_j^+ - \pi_j^- - (\pi_j^+ - \pi_j^-)^2\right)$ ……………………….. 10

Note from equations 9 and 10 that both w_j and its variance are independent of π_j^0 and hence are not affected by any possible ties between successive pairs of sampled populations.

Also, note that $\pi_j^+ - \pi_j^-$ is a measure of the difference between the probabilities that observations from population X_j are on the average greater than observations from population X_{j+1} and the probability that observations from population X_j are on the average less than observations from population X_{j+1} and is estimated by

$\hat{\pi}_j^+ - \hat{\pi}_j^- = \frac{W_j}{n} = \frac{f_j^+ - f_j^-}{n}$ ………………………………… 11

Where

$W_j = f_j^+ - f_j^-$ ……………………………………………12

Using equation $(11)\ in\ equation\ (10)$, we obtain the sample estimate of the variance of W_j as

$Var(W_j) = n(\hat{\pi}_j^+ - \hat{\pi}_j^-) - \frac{W_j^2}{n}$ ………………………………… 13

The null hypothesis that is usually of interest in time or space ordered related populations or several ordered related populations have medians M_j that are successively atmost $(or\ atleast)$ equal to each other. That is, the null hypothesis that is usually of interest is that either $M_j \leq M_{j+1}$or $M_j \geq M_{j+1}$,for$j = 1, 2, \ldots, k-1$. Hence the null hypothesis of interest here is

$H_0: \pi_j^+ \leq \pi_j^-$say versus $H_1: \pi_j^+ > \pi_j^-$ …………………… 14

Under this null hypothesis, the test statistic

$\chi_j^2 = \frac{\left(W_j - n(\pi_j^+ - \pi_j^-)\right)^2}{Var(W_j)} = \frac{\left(W_j - n(\pi_j^+ - \pi_j^-)\right)^2}{n(\pi_j^+ + \pi_j^-) - \frac{W_j^2}{n}}$ ………………………… 15 ,

For $j = 1, 2, \ldots, k-1$ has approximately the chi- square distribution with 1 degree of freedom for sufficiently large n and maybe used to test the null hypothesis of equation 14 where $\pi_j^+ - \pi_j^-$is the hypothesized difference between $\pi_j^+ and\ \pi_j^-, j = 1, 2, \ldots, k-1$. $H_0\ is\ rejected$ atα level of significance if

$c \geq \chi^2_{1-\alpha,1}$ ……………………………… 16

Otherwise, $H_0$is accepted.

However to avoid committing a type 11 Error too frequently, it is recommended that the calculated chi-square values of equation 5 be compared with the tabulated chi- square value with k-1 degrees of freedom instead of 1 degree of freedom.

Note that like W_j and its variance, the test statistic χ_j^2 of Equation 15 isnot affected by the presence of any possible ties between populations $\chi_j^2 \, and \, \chi_{j+1}^2, \; j = 1, 2, \ldots, k-1$

Of more general interest here however is that 'k' related populations that are ordered in time or space have medians that are atmost(at least) successively equal to one another. In other words, the null hypothesis that may be of interest would be that if $M_1, M_2, \ldots M_k \; are$ the medians of 'k' time or space ordered populations, then the expectation would be that M_1 is at most equal to M_2 which is in turn at most equal to M_3 and so on until M_{k-1} is at most equal to M_k say.

The sample estimate of $\sum_{j=1}^{k-1}(\pi_j^+ - \pi_j^-)$ is from equation 17

$$(\hat{\pi}_j^+ - \hat{\pi}_j^-) = \sum_{j=1}^{k-1} \frac{(f_j^+ - f_j^-)}{n} = \frac{\sum_{j=1}^{k-1} W_j}{n} = \frac{W}{n}$$

Note that like W_j and its variance which have been adjusted for the possibility of tied observations, W and its variance have also been similarly ties adjusted and are hence not affected by any ties between $X_j \; and \; X_{j+1}$ for all $j = 1, 2, \ldots, k-1$ sampled populations.

Now if pairs of the k populations that are successively ordered in time or space have at most equal medians than the difference between $\pi_j^+ \; and \; \pi_j^-$ would be expected to be equal to some constant θ_0 say which includes zero for all $j = 1, 2, \ldots, k-1$, where $\pi^+, \pi^0 \; and \; \pi^-$ are respectively the common values of $\pi_j^+, \pi_j^0 \; and \; \pi_j^- \; under \; H_0$ for $j = 1, 2, \ldots, k-1 \; and$ are estimated as

$$\hat{\pi}^+ = \sum_{j=1}^{k-1} \frac{\pi_j^+}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^+}{n(k-1)}; \; \hat{\pi}^0 = \sum_{j=1}^{k-1} \frac{\pi_j^0}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^0}{n(k-1)}; \; \hat{\pi}^- = \sum_{j=1}^{k-1} \frac{\pi_j^-}{k-1} = \sum_{j=1}^{k-1} \frac{f_j^-}{n(k-1)}; \quad ..21$$

Under the null hypothesis H_o, Equation 18 becomes $(k-1)(\hat{\pi}^+ - \hat{\pi}^-) = \frac{W}{n} \; , so \; that$

$W = n(k-1)(\hat{\pi}^+ - \hat{\pi}^-)$ ……………………..22

The sample estimate of the variance of W under H_o is then from Eqn 19

$Var(W) = n(k-1)(\hat{\pi}^+ + \hat{\pi}^- - (\hat{\pi}^+ - \hat{\pi}^-)^2)$ ………………..23

If the null hypothesis of Eqn 20 is true, then the test statistic

$\chi^2 = \frac{(W - n\theta_0)^2}{Var\,(W)} = \frac{(W - n\theta_0)^2}{n(k-1)(\hat{\pi}^+ + \hat{\pi}^- - (\hat{\pi}^+ - \hat{\pi}^-)^2)}$ ………………24

has approximately a Chi-Square distribution with k-1 degrees of freedom for sufficiently large n and may be used to test the null hypothesis H_o of Eqn 20. The null hypothesis is rejected at the α level of significance if

$\chi^2 \geq \chi^2_{1-\alpha, \; k-1}$ ……………………..25

Otherwise $H_o \; is \; accepted.$

Like the test statistic of Eqn 15, the test statistic of Eqn 24 is also unaffected by the presence of any possible ties between success ie pairs of sampled populations. The null hypothesis of Rqn20 is usually tested first. Its rejection would indicate the existence of some difference between the k population medians. In this case one would then proceed to test the null hypothesis of Eqn 14 to determine which paired populations have different medians that may have led to the rejection of the more general null hypothesis of Eqn 20.

III. Illustrative Example

Shown below are data on the letter grades earned by a random sample of 18 undergraduate students of a certain academic program during each of the five years of their studies in a university.

Student's Nos	Year1	Year2	Year3	Year4	Year 5
1	C^+	E	A^+	B	A^-
2	C^+	C	C^+	A^+	A
3	B^+	A^-	B^+	$\boldsymbol{B^+}$	B^-
4	F	C	B	C^-	B^-
5	A^-	C^-	F	F	E
6	B	B^+	B^-	E	E
7	B^-	F	A^+	A^+	B^+
8	E	B^+	A^-	B^-	C^-
9	C	B^+	B	B	F
10	C^-	C^-	E	B^-	C^+
11	C	C^-	F	C^-	A^-

12	A^{+}	B^{+}	E	C^{+}	C
13	F	C^{-}	A^{+}	F	B^{+}
14	B^{+}	E	B^{-}	B^{+}	C
15	C	C^{-}	B	B	C^{-}
16	B	A	A^{-}	B^{+}	C^{+}
17	B	A^{+}	E	A	A
18	A	A^{+}	A	A	E

To illustrate use of the proposed method, we apply Equation 2b to the above data to obtain values of u_{ij}, results of which are presented in table 1 for $i = 1, 2, \dots, 18; j = 1, 2, 3, 4$

Table 1: Tabulation of u_{ij} (Equation 2b) for the illustrative data

Student's S/No	u_{i1}	u_{i2}	u_{i3}	u_{i4}	
1	1	-1	1	-1	
2	1	-1	-1	-1	
3	-1	1	0	1	
4	-1	-1	1	-1	
5	1	1	0	-1	
6	-1	1	1	0	
7	1	-1	0	1	
8	-1	-1	1	1	
9	-1	1	0	1	
10	0	1	-1	1	
11	1	1	-1	-1	
12	1	1	-1	1	
13	-1	-1	1	-1	
14	1	-1	-1	1	
15	1	-1	0	1	
16	-1	1	1	1	
17	-1	1	-1	0	
18	-1	1	0	1	
f_j^{+}	8	10	6	10	34(= f^{+})
f_j^{0}	1	0	6	2	9(=f^{0})
f_j^{-}	9	8	6	6	29(=f^{-})
n	18	18	18	18	72(=n(k-1))
$\hat{\pi}_j^{+}$	0.444	0.556	0.333	0.556	0.472(=$\hat{\pi}^{+}$)
$\hat{\pi}_j^{0}$	0.056	0.000	0.333	0.111	0.125(=$\hat{\pi}^{0}$)
$\hat{\pi}_j^{-}$	0.500	0.444	0.333	0.333	0.389(=$\hat{\pi}^{-}$)
W_j	-1	2	2	4	7(=W)

The values of $f_{j,}^{+}$, $f_{j,}^{0}$, $f_{j,}^{-}$, $\hat{\pi}_j^{+}$, $\hat{\pi}_j^{0}$, $\hat{\pi}_j^{-}$ and W_j for j = 1, 2, 3, 4 are *calculated* as discussed above and shown in Table 1. From Equation 12, we have that W = 34-29 =5. From equation 23, we estimate the variance of 'W' as

$$Var(W) = (18)(4)(0.472 + 0.403 - (0.472 - 0.404)^2)$$

$= (72)(0.870) = 62.64$

Hence to test the null hypothesis of equation 20, we have from Equation 24, with W=5, $\theta_0 = 0$, $\text{Var}(W) = 62.640$ that

$$\chi^2 = \frac{5^2}{62.640} = 0.399, \text{which with } 5 - 1 = 4 \text{ degrees of freedom is not statistically significantlyat}$$

$\alpha = 0.05$. Hence we may conclude that students performance do not seem to be increasing (or decreasing) with time during their five years of study.

It would be instructive to compare the results obtained with the proposed methods with the results that would have been obtained if the Cochran's Q test had been used to analyse the above data. To do this, we first compare the grades of each student during every two successive years assigning the score 1 if the students grade in the past year is greater than the students grade in the immediately succeeding year and 0 otherwise for the five year period.

Application of Cochran test to the Data on letter grades of 18 students

Table 2: Relative order of the grades in Table 1 for use with Cochran Q Test

S/NO	d_{i1}	d_{i2}	d_{i3}	d_{i4}	Total $B_{i.}$
1	1	0	1	0	2
2	1	0	0	1	2
3	0	1	0	1	2
4	0	0	1	0	1
5	1	1	0	0	2
6	0	1	1	0	2
7	1	0	0	1	2
8	0	0	1	1	2
9	0	1	0	1	2
10	0	1	0	0	1
11	1	1	0	0	2
12	1	1	0	1	3
13	0	0	1	0	1
14	1	0	0	1	2
15	1	0	0	1	2
16	0	1	1	1	3
17	0	1	0	0	1
18	0	1	0	1	2
Total $T_{.j}$	8	10	6	10	34

In Table 2 d_{ij} assume the value 1 if the grade earned by the i^{th} student in year 'j' is higher than that in year $j+1$ and assumes the value 0 otherwise for $i = 1, 2, \ldots, 18$; $j = 1, 2, 3, 4$.

Now using the marginal sums $T_{.j}$ and $B_{i.}$ Shown in table 2 in the Cochran's Q test statistic, we have

$$Q = \frac{(3)\,(8^2 + 10^2 + 6^2 + 10^2 - (34)^2)/4}{34 - 61/4}$$

$$= \frac{(3)(300 - 289.0)}{34 - 15.25} = \frac{33}{18.75} = 1.76$$

which with 4 degrees of freedom is not statistically significant at $\alpha = 0.05$ thus the Cochran's Q test like the proposed test statistic is unable to reject the null hypothesis of no successive improvements (or decrease) in students performance during their five years of study. However, the proposed method unlike the Cochran's Q test would enable the researcher to quickly have a birds eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of interventionist measures.

Application of Bartholomew test to the Data on letter grades of 18 students

d_{ij}	$n_{.j}$	n	Prop(P)	Revised Prop(P')
1	8	18	0.44	0.56
2	10	18	0.56	0.56
3	6	18	0.33	0.56
4	10	18	0.56	0.56
Total				**0.56**

$$\bar{\chi}^2 = \frac{1}{\bar{p}\bar{q}} \sum_{j=1}^{n} n_{ij} \left(P_j' - \bar{P}\right)^2$$

$$\bar{\chi}^2 = \frac{1}{(0.56)(0.44)} (18(0.56 - 0.56)^2) + 18(0.56 - 0.56)^2 + 18(0.56 - 0.56)^2 + 18(0.56 - 0.56)^2$$

$= 0.000001$

Since all sample sizes are equal, m = 4, and hypothesized ordering not actually obtained in the population, the averaging process necessary before the calculation of $\bar{\chi}^2$ reduces its magnitude to insignificance at α= 0.005 level of significance. Although the proposed method and the Bartholomew approach when applied to the present data both lead to the acceptance of the null hypothesis, nevertheless the relative sizes of the calculated

Chi- square values show that the Bartholomew test statistic is more likely to lead to an acceptance of a false null hypothesis(type 1 error) more frequently and hence is likely to be less powerful than the proposed test statistic. Thus from the result of the analysis obtained, the proposed method is probably more efficient than the Bartholomew and Cochran's Q test methods.

IV. Summary And Conclusion:

This paper developed a Ties adjusted non parametric statistical method for the analysis of ordered repeated measures that are related in time, space or condition that takes account of all possible pairwise combinations of treatment levels. A test statistic is developed to determine whether subjects are increasingly performing better or worse over time or space. The proposed method unlike the Cochran's Q test and the Bartholomew's method would enable the researcher to quickly have a bird's eye view of the proportions of subject who are successively improving, experiencing no change or worsening overtime, space or condition to guide the introduction of interventionist measures.

The method is illustrated with some data and shown to be more powerful than Friedman test and shown to be easier to use than the Bartholomew procedure.

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